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# Alternative Nœtherian Banach Algebras

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## 1. Introduction

Sinclair and Tullo [6] proved that nœtherian Banach algebras are finite-dimensional. In [3], Grabiner studied nœtherian Banach modules. In this paper, we are concerned with alternative nœtherian Banach algebras. Combining techniques from [3] with techniques and the result of [6], we prove that every alternative nœtherian Banach algebra is finite-dimensional.

## 2. Preliminaries

A nonassociative algebra $A$ over a field $K$ of characteristic zero is said to be an alternative algebra if it satisfies:

$$x^2y = x(xy); \quad yx^2 = (yx)x$$

for all $x, y \in A$.

Let $A$ be an alternative algebra. $A$ is called semi-prime (respectively prime) if for any ideal $I$ of $A$ (resp. for any two of its ideals $I$ and $J$) it follows from the equality $I^2 = (0)$ (resp. $IJ = (0)$) that $I = (0)$ (resp. that either $I = (0)$ or $J = (0)$). Let $X$ be a subset of $A$. The right annihilator (respectively the left annihilator ) of $X$ in $A$ is defined by $ran(X) = \{a \in A : Xa = 0\}$ (respectively $\{a \in A : aX = 0\}$). The annihilator of $X$ is defined by $ann(X) = ran(X) \cap lan(X)$. If $A$ is semi-prime and $B$ is an ideal of $A$, then $lan(B) = ran(B) = ann(B)$ is an ideal which has zero intersection with $B$. $A$ is said to be nœtherian if it satisfies the ascending chain condition on left ideals. One can prove that in $A$, there exists a smallest ideal $B(A)$ such that $A/B(A)$ does not contain nonzero trivial ideals [7, p. 162]; $B(A)$ is called the

Baer radical of $A$. If the center $Z(A)$ of $A$ is nonzero and does not contain zero divisors of the algebra $A$ , $A$ is said to be a Cayley Dickson ring if moreover the ring of quotients $(Z(A)^*)^{-1}A$ is a Cayley Dickson algebra over the field of quotients of the center $Z(A)$ (where $Z(A)^* = Z(A) - \{0\}$).

A (real or complex) nonassociative algebra $A$ is said to be normed (respectively Banach) algebra if the underlying vector space of $A$ is endowed with a norm (respectively complete norm) $\|\cdot\|$ satisfying

$$\|ab\| \leq \|a\| \cdot \|b\|$$

for all $a, b \in A$. Any alternative algebra $A$ over a field $K$ can be imbedded in a unital alternative algebra $A'$

$$A' = K + A$$

For basic results on alternative algebras, the reader is referred to [7]. In particular, recall that every prime alternative algebra $A$ that is not associative is a Cayley Dickson ring. Further, the Baer radical of a nœtherian alternative algebra is nilpotent. Finally, note that if $A$ is an alternative algebra, for any two of its ideals $I$ and $J$, the product $IJ$ is also an ideal of the algebra $A$.

## 3. Main result

Given a nonassociative algebra $A$ over a field $K$, the left multiplication algebra $L(A)$ of $A$ is defined to be the subalgebra of $End_K(A)$ generated by all left multiplications $L_a$, $a$ in $A$, and the identity $Id$ on $A$. If $A$ is a normed algebra then $L(A)$ is clearly a subalgebra of the normed associative algebra $BL(A)$ of bounded linear operators of $A$. In this case, the closed left multiplication algebra is defined to be the closure $L(A)^-$ of $L(A)$ in $BL(A)$.

LEMMA 1. *Let $A$ be a nonassociative Banach algebra over $K$ ($K = \mathbb{R}$ or $K = \mathbb{C}$) and let $y$ be an element of the nucleus $N(A)$ of $A$. If the closure $(A'y)^-$ of $A'y$ is finitely generated as a left ideal in $A$ then $A'y$ is closed.*

*Proof.* Note first that for $y$ in $N(A)$, $b$ in $A$, and $T$ in $L(A)^-$, we have

$$TL_by = L_{Tb}y. \tag{1}$$

By linearity and continuity we need only to consider the case that $T = T_n = L_{a_1} \cdots L_{a_n}$ with $a_i$ in $A$, $1 \leq i \leq n$. We will induct on $n$.

For $n = 1$, (1) holds since

$$L_{a_1} L_b y = a_1(by) = (a_1 b)y = L_{L_{a_1} b} y,$$

because $y$ lies in the nucleus.

Suppose now that (1) is true for all $m \leq n-1$. Then

$$T_n L_b y = L_{a_n}(T_{n-1} L_b y) = L_{a_n}(L_{T_{n-1} b} y) = L_{L_{a_n} T_{n-1} b} y = L_{T_n b} y,$$

as required.

Suppose now that the closure $(A'y)^-$ of the left ideal $A'y$ is finitely generated as a left ideal. Then

$$(A'y)^- = L(A)x_1 + \cdots + L(A)x_n = L(A)^- x_1 + \cdots + L(A)^- x_n,$$

for some $x_1, \ldots, x_n$ in $(A'y)^-$ because $(A'y)^-$ is closed. For each $1 \leq i \leq n$, choose a sequence $\{a_{ik}\}_k$ in $A'$ such that $\{a_{ik} y\}_k$ converges to $x_i$. Let $f_k$ be the map

$$\begin{array}{rccc} f_k : & L(A)^- \times \cdots \times L(A)^- & \longrightarrow & (A'y)^- \\ & (T_1, \ldots, T_n) & \rightsquigarrow & T_1(a_{1k} y) + \cdots + T_n(a_{nk} y), \end{array}$$

and let $f$ be the map given by $f(T_1, \ldots, T_n) = T_1(x_1) + \cdots + T_n(x_n)$. The sequence $\{f_k\}$ converges uniformly to $f$ on $L(A)^- \times \cdots \times L(A)^-$ but the set of surjective continuous linear operators is open [2]. Hence there exists a positive integer $k$ such that $f_k$ is surjective. Now, by (1),

$$\begin{aligned} f_k(T_1, \ldots, T_n) &= T_1 L_{a_{1k}} y + \cdots + T_n L_{a_{nk}} y \\ &= L_{T_1(a_{1k})} y + \cdots + L_{T_n(a_{nk})} y \\ &= L_{(T_1 a_{1k} + \cdots + T_n a_{nk})} y \in A'y. \end{aligned}$$

Thus $(A'y)^- = A'y$, as required. ∎

LEMMA 2. *Let $A$ be a nonassociative complex Banach algebra which satisfies the ascending chain condition on left ideals. Assume that the center $Z(A)$ of $A$ consists of regular elements. Then $Z(A) \simeq \mathbb{C}$.*

*Proof.* Suppose that some $x \in Z(A)$ has infinite spectrum in $Z(A)$ then the boundary of the spectrum of $x$, $\partial(sp(x, Z(A)))$ contains an infinite sequence $\{\lambda_n\}$ of distinct nonzero complex numbers. For each positive integer $n$ define

$$I_n = \Big\{ z \in A,\ z(x - \lambda_1) \cdots (x - \lambda_n) = 0 \Big\}.$$

$\{I_n\}$ is an increasing sequence of left ideals in $A$. Define

$$\begin{array}{rcl} T : A' & \longrightarrow & A' \\ y & \rightsquigarrow & yx. \end{array}$$

We check easily that $sp(T) = sp(x, Z(A))$. Thus $\lambda_n \in \partial(spT)$. By Lemma 1, $A'(x - \lambda_n) = Im(T - \lambda_n)$ is closed. So, $\lambda_n$ is an eigenvalue of $T$ [5]. Each $\lambda_n$ eigenvector of $T$ is in $I_n$ but not in $I_{n-1}$ so that $\{I_n\}$ is a strictly increasing sequence of left ideals in $A$ contrary to hypothesis. Thus, each element of $Z(A)$ has finite spectrum.

Let $x$ be in $RadZ(A)$ and consider

$$\begin{array}{rcl} T : A' & \longrightarrow & A' \\ y & \rightsquigarrow & yx, \end{array}$$

$x$ is quasi-nilpotent, so $sp(T) = \{0\}$. Applying again Lemma 1 and [5, VII, Propositions 6.4 and 6.7] we deduce that there exists $z \in A$ such that $zx = 0$. Then, $x = 0$ by hypothesis. And thus, $Z(A)$ is semi-simple. Consequently, $Z(A)$ is finite-dimensional [4]. Hence, $Z(A)$ is isomorphic to the complex field by the Wedderburn theorem for semi-simple finite-dimensional associative complex algebras. ∎

As a consequence of lemma 2 and Slater's theorem for prime nondegenerate alternative algebras [7, p. 194], we obtain:

LEMMA 3. *Let $A$ be a complex nœtherian alternative prime Banach algebra which is not associative. Then $A = \mathbb{O}_{\mathbb{C}}$ (the Cayley Dickson algebra over $\mathbb{C}$).*

THEOREM 4. *Let $A$ be an alternative nœtherian complex Banach algebra. Then $A$ is finite-dimensional.*

*Proof.* In the prime case this follows from Lemma 3 and the corresponding result for nœtherian associative Banach algebras [6]. Suppose now that $A$ is semi-prime. We claim that $A$ can be embedded in a direct product of a finite number of prime alternative nœtherian Banach algebras, and hence $A$ would be finite-dimensional by the previous prime case.

To prove the claim, let $A$ be a semi-prime alternative nœtherian Banach algebra. We note first that $A$ satisfies both acc and dcc on annihilator ideals because the first annihilator coincides with the third one. Denote by $\mathfrak{I}$ the family of all nonzero ideals $M$ of $A$ such that $ann(M)$ is maximal in the set of all annihilator ideals $ann(B)$, where $B$ is a nonzero ideal of $A$. Now we have:

i) For any $M \in \Im$, $ann(M)$ is a prime ideal of $A$.

ii) Any nonzero annihilator ideal $ann(I)$, $I$ being an ideal of $A$, contains an ideal $M \in \Im$.

iii) There exist finitely many ideals $M_1, \dots, M_n$ in $\Im$ such that $ann(M_1 + \cdots + M_n) = \bigcap ann(M_i) = 0$.

(i) Let $B$ be an ideal of $A$ containing strictly $ann(M)$. Then $B \cap M$ is nonzero and hence $ann(B \cap M) = ann(M)$ by maximality of $ann(M)$. Now if $C$ is another ideal of $A$ containing $ann(M)$, $BC \subseteq ann(M)$ implies $(M \cap B)C \subseteq ann(M) \cap M = 0$. Hence, $C$ is contained in $ann(B \cap M) = ann(M)$. Therefore $A/ann(M)$ is prime.

(ii) By the acc on annihilator ideals, there exists a nonzero ideal $N$ of $A$ contained in $ann(I)$ such that $ann(N)$ is maximal in the set of the annihilator ideals $ann(B)$, $B$ a nonzero ideal of $A$ contained in $ann(I)$; but $ann(N)$ is actually maximal in the set of all annihilator ideals. Indeed, let $C$ be a nonzero ideal of $A$ such that $ann(N) \subseteq ann(C)$. Then $C$ is contained in $ann(ann(C))$ and hence in $ann(ann(N))$, because annihilator reverse inclusions, but $N \subseteq ann(I)$ implies $ann(ann(I)) \subseteq ann(N)$ and hence $ann(ann(N))$ is contained in $ann(ann(ann(I))) = ann(I)$. Then $C$ is actually contained in $ann(I)$. Hence $ann(N) = ann(C)$ which implies that $N \in \Im$.

(iii) Let $I = \sum M_i$ where $M_i$ ranges over $\Im$. Since $A$ is nœtherian, $I$ is generated by a finite number of $M_i$, that is, $I = M_1 + \cdots + M_n$. Now $ann(I) = \cap ann(M_i) = 0$, since otherwise $ann(I)$ would contain an ideal $M \in \Im$ by (ii), and hence $M$ would be contained in $I \cap ann(I) = 0$ by semiprimeness of $A$, which is a contradiction.

Therefore $A$ is a subdirect product of the alternative nœtherian Banach algebras $A/ann(M_i)$, $1 \leq i \leq n$, each of which is prime by $(i)$, which concludes the proof of the claim.

Consider now the general case. Let $B$ be the Baer radical of $A$. By [7, Theorem 5, p. 256], $B$ is nilpotent ideal (with index of nilpotence, say $n$) containing any solvable, in particular nilpotent ideal of $A$. Since the closure of $B$ is also nilpotent with the same index of nilpotence, $B$ is closed and $A/B$ is a semi-prime alternative nœtherian Banach algebra, and therefore finite dimensional. Consider the following sequence of ideals $B^j$ defined inductively by $B^1 = B$ and $B^{j+1} = BB^j$ (recall that the product of two ideals of an alternative algebra is an ideal). Then $B^j/B^{j+1}$ can be regarded as a finitely generated left $L(A/B)$-module. Since $A/B$ is finite dimensional, $L(A/B)$ is also finite dimensional, and hence the same is true for $B^j/B^{j+1}$. In particular,

$B^{n-1}$ is finite dimensional since $B^n = 0$. A recursive argument allows then us to show that $B$ is finite dimensional, which completes the proof. ∎

COROLLARY 5. *Let $A$ be a real alternative nœtherian Banach algebra, then $A$ is finite-dimensional.*

*Sketch of the proof:* For the proof, we prove first that $A_{\mathbb{C}}$ is nœtherian. By applying Theorem 4, we deduce that $A_{\mathbb{C}}$ and hence $A$ is finite-dimensional.

*Remark.* Sidney has shown that a Banach algebra in which all left ideals are closed is nœtherian (see for example [1] for the proof). Using the similar argument, we can prove that an alternative Banach algebra is nœtherian if and only if all its left ideals are closed.

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## REFERENCES

[1] AUPETIT, B., "Propriétés Spectrales de Algèbres de Banach", Lecture Notes in Math., Vol. 735, Springer-Verlag, Berlin-New York, 1979.

[2] GLEASON, A.M., Finitely generated ideals in Banach algebras, *J. Math. Mech.*, **13** (1964), 125-132.

[3] GRABINER, S., Finitely generated nœtherian and artinian Banach algebras, *Indiana Univ. Math. J.*, **26** (1977), 413-425.

[4] KAPLANSKY, I., Ring isomorphisms of Banach algebras, *Canad. J. Math.*, **6** (1954), 374-381.

[5] CONWAY, J.B., "A Course in Functional Analysis", Graduate Texts in Math., Vol. 96, Springer-Verlag, New York-Berlin, 1985.

[6] SINCLAIR, A.M., TULLO, A.W., Nœtherian Banach algebras are finite dimensional, *Math. Ann.*, **211** (1974), 151-153.

[7] ZHEVLAKOV, A., SLIN'KO, A.M. SHESTAKOV, I.P., SHIRSHOV, A.I., "Rings that are Nearly Associative", Pure Appl. Math., Vol. 104, Academic Press, New York-London, 1982.