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Generalization of Projection Constants: Sufficient Enlargements*

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1. Introduction and main definitions

Let X be a Banach space and let Y be a finite dimensional subspace. We denote the unit ball of X by $B(X)$. Let $P\colon X \to Y$ be some continuous linear projection. Then $P(B(X)) \supset B(Y)$ and $P(B(X))$ is a convex, symmetric with respect to 0, bounded subset of Y. It is very natural to consider the following question: How small can be made the set $P(B(X))$ under a proper choice of P? It is clear that the word "small" has a vague meaning in this context.

For one of the possible meanings of the word "small" this question was investigated in very many papers, I mean investigations of projection constants. But to the best of my knowledge for other notions of "smallness" the question was not considered.

In order to investigate the question it is natural to introduce the following definition.

Let X be a finite dimensional normed space.

Definition 1. A symmetric with respect to 0, bounded, closed convex body $A \subset X$ will be called a *sufficient enlargement* for X (or of $B(X)$) if for arbitrary isometric embedding $X \subset Y$ there exists a projection $P\colon Y \to X$ such that $P(B(Y)) \subset A$.

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Convention. We shall use the term *ball* for symmetric with respect to 0, bounded, closed convex body with nonempty interior in a finite dimensional linear space.

Remark. It is clear that we have the following identity for the absolute projection constant

$$\lambda(X) = \inf\{\lambda \in \mathbb{R}^+ : \lambda \cdot B(X) \text{ is a sufficient enlargement for } X\}.$$

We use standard definitions and notation of Banach space theory (see [5], [12]).

2. Some examples and observations

Let A be a ball in a finite dimensional space X. The space X normed by the gauge functional of A will be denoted by X_A.

We start with some simple observations. Their proofs are straightforward and we omit them. By γ_∞ we denote the L_∞–factorable norm (see [12], (p. 95)).

Proposition 1. *A ball A is a sufficient enlargement for X if and only if $\gamma_\infty(I) \leq 1$, where I is the natural identity mapping $I \colon X \to X_A$.*

Corollary 1. *If X and Y are $\mathbb{R}^n$ with different norms and $B(X) \subset B(Y)$ then every sufficient enlargement for Y is a sufficient enlargement for X.*

Corollary 2. *Let $T \colon X \to Z$ be an invertible linear operator between finite dimensional normed spaces. Then*

$$\gamma_\infty(T) \cdot T^{-1}(B(Z))$$

is a sufficient enlargement for X.

Remark 1. Of course the statement of Proposition 1 remains true if we replace $\gamma_\infty(T)$ by a greater norm on the space of operators, e.g. by $\pi_2(T)$.

This observation is useful for example for those spaces X for which there exist good estimates of $\pi_2(T)$ for some operators $T \colon X \to l_2$. In particular, by the well-known fact $\pi_2(I \colon l_1^n \to l_2^n) = 1$ (see e.g. [8]), $B(l_2^n)$ is a sufficient enlargement for l_1^n for every $n \in \mathbb{N}$. The analogous assertion is true for "most" of random n-dimensional quotients of l_1^{2n} (see [8] and [11]).

Remark 2. Estimates of γ_∞–norms of natural embeddings $l_p^n \to l_q^n$ see in [2].

COROLLARY 3. *A symmetric with respect to* 0 *parallelepiped containing* $B(X)$ *is a sufficient enlargement for* X.

PROPOSITION 2. [4] *Convex combination of sufficient enlargements for* X *is a sufficient enlargement for* X.

The same is true for integrals with respect to probability measures. In order to make this statement precise we need to introduce a notion of integral of function, whose values are convex subsets in $\mathbb{R}^n$.

I introduce the notion of integral for convex body–valued functions as some mixture of Riemann and Lebesgue integrals. This definition of integral is somewhat unnatural, but it is sufficient for our purposes and at the moment I do not want to overcome difficulties which appear for more general notions of integral.

Let M be a compact metric space with a regular Borel probability measure μ. (The main example for us is the group of orthogonal matrices in $\mathbb{R}^n$ or its closed subgroups with the normalized Haar measures).

The set of all compact convex subsets of $\mathbb{R}^n$ will be denoted by $C(n)$. We shall consider $C(n)$ as a metric space with the Hausdorff metric:

$$d(A,B) = \max\{\sup_{a\in A} \operatorname{dist}(a,B),\ \sup_{b\in B} \operatorname{dist}(b,A)\}.$$

Recall the following well-known fact: $C(n)$ is complete with respect to d. For this and other results on convex bodies we refer to [9].

Let $f\colon M \to C(n)$ be a continuous function.

DEFINITION 2. The integral of f with respect to measure μ is defined to be

$$\int_M f(m)d\mu(m) := \lim_{diam\Delta\to 0} \sum_{i=1}^{k(\Delta)} f(a_i(\Delta))\mu(M_i(\Delta)), \tag{1}$$

where Δ is a pair consisting of a partition of M onto a finite number of measurable subsets $\{M_i(\Delta)\}_{i=1}^{k(\Delta)}$ and a family $\{a_i(\Delta)\}_{i=1}^{k(\Delta)}$ of points for which $a_i(\Delta) \in M_i(\Delta)$. Diameter of Δ is defined to be the maximum of the diameters of the sets $M_i(\Delta)$ $(i = 1, \dots, k(\Delta))$ in the metric space M. The limit in (1) is considered in the Hausdorff metric.

A proof that the integral exists can be obtained in the same way as the proof of existence of Riemann integral in classical analysis.

PROPOSITION 3. [4] *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a normed space and M be a compact metric space with a probability measure μ. Suppose that a mapping $f\colon M \to C(n)$ is continuous and that $f(m)$ is a sufficient enlargement for X for all $m \in M$. Then*

$$\int_M f(m)\, d\mu(m)$$

is also a sufficient enlargement for X.

Remark. B.Grünbaum used Proposition 3 in order to find the precise upper estimate for $\lambda(l_2^n)$.

3. SUFFICIENT ENLARGEMENTS AND INTEGRALS OF PARALLELEPIPEDS

Corollary 3 and Propositions 2 and 3 supply us with the following family of sufficient enlargements for a space X: parallelepipeds containing $B(X)$, their convex combinations and integrals with respect to probability measures. It is natural to ask: is it true that any sufficient enlargement contains some sufficient enlargement of the described type?

The answer to this question is negative. The first example was found by V.M.Kadets (1993). In his example X is a two-dimensional space, whose unit ball is a regular hexagon. The space X can be isometrically embedded into l_∞^3. Let $P\colon l_\infty^3 \to X$ be the orthogonal projection. It is clear that $A := P(B(l_\infty^3))$ is a sufficient enlargement for X. V.M.Kadets proved that A does not contain any integral with respect to a probability measure of parallelograms containing $B(X)$.

Our purpose is to prove that analogous examples can be constructed even for two dimensional Euclidean space.

THEOREM. *There exists a sufficient enlargement for l_2^2, which does not contain any integral with respect to a probability measure of parallelograms containing $B(l_2^2)$.*

Proof. Let us denote by S_1 and S_2 the operators of counterclockwise rotation of l_2^2 onto $2\pi/3$ and $4\pi/3$ respectively. Let e_1 and e_2 be the unit vector basis of l_2^2 and e_1^* and e_2^* be its biorthogonal functionals.

It is easy to verify that for all $x, y \in \mathbb{R}^2$, $\|y\|_2 = 1$ we have

$$x = \frac{2}{3}(\langle x, y\rangle y + \langle x, S_1 y\rangle S_1 y + \langle x, S_2 y\rangle S_2 y).$$

Let $y = e_2$. We have the following factorization of the identity operator on l_2^2:

$$I = RQ,\ l_2^2 \stackrel{Q}{\to} l_\infty^3 \stackrel{R}{\to} l_2^2,$$

where

$$Q(x) = \{\langle x, e_2\rangle, \langle x, S_1 e_2\rangle, \langle x, S_2 e_2\rangle\},$$

$$R(\{a_0, a_1, a_2\}) = \frac{2}{3}(a_0 e_2 + a_1 S_1 e_2 + a_2 S_2 e_2).$$

Hence the Minkowski sum of the line segments

$$A = \frac{2}{3}([-e_2, e_2] + [-S_1 e_2, S_1 e_2] + [-S_2 e_2, S_2 e_2])$$

is a sufficient enlargement for l_2^2.

It is easy to verify that A is a regular hexagon with

$$\sup\{e_1^*(x) :\ x \in A\} = \frac{2}{\sqrt{3}}.$$

We need the following lemma.

LEMMA. *Let P be a parallelogram containing $B(l_2^2)$. Then*

$$\sup\{e_1^*(x) :\ x \in \frac{1}{3}(P + S_1 P + S_2 P)\} > \frac{2}{\sqrt{3}}.$$

Proof. We represent P as a sum of two line segments: $P = [-f_1, f_1] + [-f_2, f_2]$.

We introduce the notation

$$a := \sup\{e_1^*(x) :\ x \in \frac{1}{3}(P + S_1 P + S_2 P)\}.$$

We have

$$a = \frac{1}{3}(|e_1^*(f_1)| + |e_1^*(f_2)| + |e_1^*(S_1 f_1)| + |e_1^*(S_1 f_2)| + |e_1^*(S_2 f_1)| + |e_1^*(S_2 f_2)|).$$

Set

$$t(f_1) := \frac{1}{3}(|e_1^*(f_1)| + |e_1^*(S_1 f_1)| + |e_1^*(S_2 f_1)|).$$

Let us show that

$$t(f_1) \geq \frac{\|f_1\|}{\sqrt{3}},$$

and the equality is attained if and only if the angle between f_1 and e_2 is a multiple of $\pi/3$.

It is easy to see that in order to prove this statement it is sufficient to consider the case when the angle α between f_1 and e_2 is in the interval $[0, \frac{\pi}{3}]$.

We have

$$\begin{aligned} t(f_1) &= \frac{\|f_1\|}{3}(|\sin\alpha| + |\sin(\alpha + \frac{2\pi}{3})| + |\sin(\alpha + \frac{4\pi}{3})|) \\ &= \frac{\|f_1\|}{3}(\sin\alpha + \sin(\alpha + \frac{2\pi}{3}) - \sin(\alpha + \frac{4\pi}{3})) \\ &= \frac{\|f_1\|}{3}(\sin\alpha + \sqrt{3}\cos\alpha). \end{aligned}$$

It is clear that for vectors of the same norm this product is minimal if and only if $\alpha = 0$ or $\alpha = \pi/3$. In both cases we have $t(f_1) = \|f_1\|/\sqrt{3}$. So we have proved the assertion about $t(f_1)$.

Since $a = t(f_1) + t(f_2)$, then

$$a \geq \frac{\|f_1\| + \|f_2\|}{\sqrt{3}},$$

and the equality is attained if and only if the angles between f_1, f_2 and e_2 are multiples of $\pi/3$. On the other hand since $[-f_1, f_1] + [-f_2, f_2] \supset B(l_2^2)$, then $\|f_1\|, \|f_2\| \geq 1$ and if the angles between f_1, f_2 and e_2 are multiples of $\pi/3$, then

$$\|f_1\| + \|f_2\| > 2.$$

Hence $a > 2/\sqrt{3}$. ∎

We return to the proof of the theorem. Suppose the contrary. Let M be a metric space with a probability measure μ and let $F\colon M \to C(n)$ be a uniformly continuous function for which $F(m)$ is a parallelogram containing $B(l_2^2)$ for each $m \in M$ and

$$\int_M F(m)\, d\mu(m) \subset A.$$

Since A is invariant under action of S_1 and S_2, then

$$\int_M \frac{1}{3}(F(m) + S_1F(m) + S_2F(m))d\mu(m) \subset A. \tag{2}$$

Hence

$$\sup\{e_1^*(x):\ x \in \int_M \frac{1}{3}(F(m)+S_1F(m)+S_2F(m))\,d\mu(m)\} \le \frac{2}{\sqrt{3}}.$$

This supremum equals to

$$\int_M \sup\{e_1^*(x):\ x \in \frac{1}{3}(F(m)+S_1F(m)+S_2F(m))\}\,d\mu(m).$$

By the lemma the integrand is $> \frac{2}{\sqrt{3}}$ for each m. Hence the integral is $> \frac{2}{\sqrt{3}}$. This contradicts (2). ▮

It is natural to consider an "isomorphic" version of the question above. I mean the following. If a sequence $\{X_n\}_{n=1}^{\infty}$ of finite dimensional normed spaces is such that for some sufficient enlargements A_n ($n \in \mathbb{N}$) for X_n, arbitrary $0 < C < \infty$ and arbitrary integrals I_n with respect to probability measures of parallelepipeds containing $B(X_n)$ we have

$$\exists n \in \mathbb{N},\ I_n \not\subseteq C \cdot A_n,$$

then we shall say that $\{X_n\}$ has *property N*.

PROBLEM 1. Do there exist sequences $\{X_n\}$ with property N?

PROBLEM 2. Does the sequence $\{l_2^n\}_{n=1}^{\infty}$ have property N?

PROBLEM 3. Does every sequence $\{X_n\}_{n=1}^{\infty}$ for which Banach–Mazur distances

$$d(X_n, l_\infty^n)$$

go to ∞ have property N?

Remark. One of the natural approaches to Problem 2 is to consider multiple tensor products of the construction of the theorem.

4. Some other problems on sufficient enlargements

It seems that the most important direction of research connected with the sufficient enlargements is the following.

Clearly, for a space with large absolute projection constant, sufficient enlargements should be much larger than the unit ball in some directions. The problem is to describe the corresponding sets of directions. The problem can be divided in a natural way into the following two problems.

PROBLEM 4. What is the size of this set (for any reasonable notion of size)?

PROBLEM 5. What are the restrictions on the shape of this set of directions?

Remark. In connection with these problems it is worthwhile to mention that K.Ball [1] proved that for every ball B the ellipsoid E of maximal volume in B and the parallelepiped Q of minimal volume containing B satisfy

$$(\mathrm{vol}Q/\mathrm{vol}E)^{1/n} \leq \sqrt{e}(\mathrm{vol}B(l_\infty^n)/\mathrm{vol}B(l_2^n))^{1/n}.$$

The results of [7] and [10] may also be of interest in this connection.

Remark. Investigations of large subspaces of l_∞^n (see Example after Corollary 8 in [3]) show that the n-th root of the volume of sufficient enlargement can be much less than the n-th root of the volume of $\lambda(X)B(X)$.

Some additional information on sufficient enlargements can be found in [6].

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