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# A Note on Hacque's Cohomology of Rings-Groups and Extensions of Rings-Groups by Groups

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## Introduction

The purpose of Hacque in [12] is developing an environment in which the classical theory of group extensions and a general theory of ring extensions (or equivalently a general theory of crossed products of rings by groups) can be stated as particular cases. Then he introduces the category of *rings-groups* and he defines the concept of extension of rings-groups by groups. In order to classify such extensions through a definition of crossed products of groups by rings-groups, he needs to introduce a 2-dimensional cohomology of groups with coefficients in rings-groups, that is done in his paper [11]. But there is already a non-abelian 2-dimensional cohomology of groups, which was first defined by Dedecker [5] and which takes crossed modules as coefficients. The main purpose of this note is to connect both cohomologies. Then we associate a crossed module $\Phi(\Gamma)$ to a ring-group $\Gamma$ and we see how Hacque's non-abelian cohomology $\widetilde{H}^2(G,\Gamma)$ coincides with Dedecker's $\mathbf{H}^2(G,\Phi(\Gamma))$ at the underline set level. After this identification, we can use the proved results for Dedecker's cohomology, as for example:

- $\mathbf{H}^2(G,\Phi)$ classifies non-singular extensions of $G$ by the crossed module $\Phi$, see [7].
- $\mathbf{H}^2(G,\Phi)$ has a simplicial interpretation and it is a cotriple cohomology, see [3].

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- There is an obstruction theory for Dedecker's cohomology, see [4] or [3].
- There are six term exact sequences for Dedecker's cohomology, associated to short exact sequences of crossed modules (Dedecker's $\mathbf{H}^2$ behaves as the first derived of the non-abelian Hom functor), see [8].
- There are Hochschild-Serre's type sequences using Dedecker's cohomology, see [2].

These results could make Hacque's work easier and moreover they can help to develop some of his projects.

## 1. Dedecker's Non-Abelian Cohomology with Coefficients on Crossed Modules

Let us recall that a crossed module

$$\Phi = \left(M \xrightarrow{\rho} N, N \xrightarrow{\Psi} \mathrm{Aut}(M)\right)$$

consists of a pair of group homomorphisms $\rho : M \to N$ and $\Psi : N \to \mathrm{Aut}(M)$ which satisfy the following two conditions

(i) $\rho({}^{n}m) = n\rho(m)n^{-1}$,
(ii) ${}^{\rho(m)}m' = mm'm^{-1}$,

for all $m, m' \in M$ and $n \in N$, where ${}^{n}m$ denotes $\Psi(n)(m)$. Let us also note that the morphism $\Psi$ is equivalent to an $N$-group structure on $M$, therefore a crossed module is usually written just as $\Phi : M \xrightarrow{\rho} N$. A morphism $\gamma : \Phi \to \Phi'$ of crossed modules is a commutative diagram

$$\begin{array}{cccc} \Phi : & M & \xrightarrow{\rho} & N \\ {\scriptstyle \gamma=(\gamma_1,\gamma_0)}\downarrow & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_0} \\ \Phi' : & M' & \xrightarrow{\rho'} & N' \end{array}$$

such that

$$\gamma_1({}^{n}m) = {}^{\gamma_0(n)}\gamma_1(m),$$

for all $n \in N$ and $m \in M$. Let $\mathcal{XM}$ be the corresponding category of crossed modules.

Given a group $G$ and a crossed module $\Phi$ as above, Dedecker defines a 2-*cocycle* of $G$ with coefficients in $\Phi$ as a pair of maps $(\eta : G \to N, \mu : G \times G \to M)$ satisfying the cocycle conditions:

$$\eta_\alpha \eta_\beta = \rho(\mu_{\alpha,\beta})\eta_{\alpha\beta}, \tag{1}$$

and

$$\eta_\alpha(\mu_{\beta,\gamma})\mu_{\alpha,\beta\gamma}\mu_{\alpha\beta,\gamma}^{-1}\mu_{\alpha,\beta}^{-1} = 1, \tag{2}$$

for all $\alpha, \beta, \gamma \in G$. Two 2-cocycles $(\eta, \mu)$ and $(\eta', \mu')$ are *equivalent* (or cohomologous) if there exists a map $a : G \to M$ such that

$$\eta'_\alpha = \rho(a_\alpha)\eta_\alpha \qquad \text{and} \qquad \mu'_{\alpha,\beta} = a_\alpha{}^{\eta_\alpha}a_\beta\mu_{\alpha,\beta}a_{\alpha\beta}^{-1}$$

for all $\alpha, \beta \in G$. This defines an equivalence relation and the set $\mathbf{H}^2(G, \Phi)$ of equivalence classes of 2-cocycles of $G$ with coefficients in $\Phi$ is by definition Dedecker's 2-dimensional non abelian cohomology. This set $\mathbf{H}^2(G, \Phi)$ has a subset $\mathbf{O}^2(G, \Phi)$ of distinguished elements which are called *neutral* elements and which are those equivalent classes of cocycles with a representative element in the form $(0, \mu)$.

This non-abelian 2-cohomology introduced by Dedecker in the later fifties has very nice properties. For example, it is functorial in both variables and any morphism of groups (or crossed modules) takes neutral elements to neutral ones (see [4]). It has also an interpretation in terms of equivalent classes of non-singular group extensions (see also [4]): A non-singular group extension of $G$ by $\Phi$ (or just $\Phi$-extension of $G$) is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle v} & & & & \\ & & M & \xrightarrow{\rho} & N & & & & \end{array} \tag{3}$$

where the top row is a short exact sequence of groups ($M$ is normal subgroup of $E$ and the quotient $E/M$ is $G$) and the action of $E$ on $M$ given by conjugation coincides with that induced by $v$. Let us note that any $\Phi$-extension of $G$ induces a group homomorphism $\theta : G \to \text{coker}(\rho)$. The set of equivalent classes of $\Phi$-extensions of $G$ (with the natural definition of equivalence) is denoted by $\text{Ext}(G, \Phi)$ and it is the disjoint union of the sets $\text{Ext}_\theta(G, \Phi)$ of equivalence classes of $\Phi$-extension which induce a given $\theta$, where $\theta$ moves on the set of group homomorphisms from $G$ to $\text{coker}(\rho)$.

Then there is a natural bijection

$$\text{Ext}(G, \Phi) \overset{\varphi}{\cong} \mathbf{H}^2(G, \Phi),$$

where the image of the equivalence class of a non-singular extension as in (3) is the cohomology class of the cocycle $(\eta, \mu)$ with

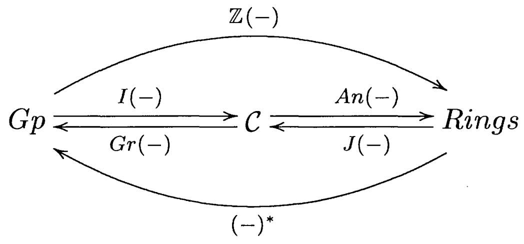
$$\eta_\alpha = \upsilon(u_\alpha)$$
$$\mu_{\alpha,\beta} = u_\alpha u_\beta u_{\alpha\beta}^{-1}$$

for any $\alpha, \beta \in G$, and $u : G \to E$ a section (in Set) of $p$, see [7] or [3]. Moreover, any cohomology class of 2-cocycles in $\mathbf{H}^2(G, \Phi)$ has also associated a group homomorphism $\theta : G \to \operatorname{coker}(\rho)$ and the set $\mathbf{H}^2(G, \Phi)$ is the disjoint union of the sets $\mathbf{H}^2_\theta(G, \Phi)$ of cohomology classes of cocycles which induces $\theta$, also the above isomorphism $\varphi$ induces isomorphisms $\operatorname{Ext}_\theta(G, \Phi) \cong \mathbf{H}^2_\theta(G, \Phi)$.

## 2. Hacque's Non-Abelian Cohomology with Coefficients in Rings-Groups

Let us now recall Hacque's cohomology of groups with coefficients in rings-groups.

A ring-group $\Gamma = [V, M]$ consists of a ring $V$ together with a subgroup $M \subseteq V^*$ of the group $V^*$ of invertible elements of $V$. A morphism $\phi : \Gamma = [V, M] \to \Gamma' = [V', M']$ of rings-groups is a ring homomorphism $\phi : V \to V'$ such that $f(M) \subseteq M'$. The category of rings-groups will be denoted by $\mathcal{C}$, then there is a commutative diagram of categories and functors




where the functors on the top places are left adjoint to those setting on the analogous bottom places, and where $Gp$ and $Rings$ are the categories of groups and rings respectively, $\mathbb{Z}(-)$ and $(-)^*$ are the canonical functors which takes a group $G$ to the integer group-ring of $G$ and for any ring $R$, $(R)^* = R^*$ is the multiplicative group of invertible elements of $R$, the functor $Gr(-)$ takes a ring-group $\Gamma = [V, M]$ to $M$ and $An(\Gamma) = V$. Moreover, the functors $I(-)$ and $J(-)$ are both full and faithful and so we can identify the categories of groups and rings as full reflexive and coreflexive subcategories of $\mathcal{C}$ respectively, see [11].

Any ring group $\Gamma = [V, M]$ has associated an exact sequence of groups

$$1 \to \mathcal{Z}_g(\Gamma) \to Gr(\Gamma) \stackrel{\omega}{\to} \mathrm{Aut}(\Gamma) \to \mathrm{Aut}_e(\Gamma) \to 1$$

where $\mathrm{Aut}(\Gamma)$ is the group of automorphisms of $\Gamma$ in $\mathcal{C}$, and it consists of those ring automorphisms $\phi : V \to V$ such that $\phi(M) = M$, the homomorphism $\omega$ takes any element $a \in M$ to the inner automorphisms of $V$ given by conjugation by $a$, which will be denoted by $< a >$. $\mathcal{Z}_g(\Gamma)$ is an abelian group called the group *center* of $\Gamma$ and $\mathrm{Aut}_e(\Gamma)$ is the quotient group of $\mathrm{Aut}(\Gamma)$ by the normal subgroup of inner automorphisms and it is called the group of *outer automorphisms* of $\Gamma$ (see [11]). It is clear that the morphism $Gr(\Gamma) \stackrel{\omega}{\to} \mathrm{Aut}(\Gamma)$ together with the canonical action of $\mathrm{Aut}(\Gamma)$ on $Gr(\Gamma)$ is a crossed module which we will denote by $\Phi(\Gamma)$. Let us note that the correspondence $\Gamma \mapsto \Phi(\Gamma)$ is not functorial.

Now, for any group $G$ and any ring-group $\Gamma$ the group $\widehat{C}^1(G, \Gamma)$ of *generalized weak* 1-*cochains* consists of pairs of normalized maps $(a : G \to M, b : G \times G \to \mathcal{Z}_g(\Gamma))$ (we will denote by $a_\alpha$ the image of $\alpha \in G$ by $a$). If $b$ is the zero map then the pair $(a, 0)$ is called a (generalized) strict 1-cochain and the subgroup of strict 1-cochains of $\widehat{C}^1(G, \Gamma)$ will be denoted by $\tilde{C}^1(G, \Gamma)$. On the other hand, the $\widehat{C}^1(G, \Gamma)$-set $\widehat{C}^2(G, \Gamma)$ of (generalized) 2-cochains consists of pairs of normalized maps $(\eta : G \to \mathrm{Aut}(\Gamma), \mu : G \times G \to M)$ which satisfies the equation

$$\eta_\alpha \eta_\beta =< \mu_{\alpha,\beta} > \eta_{\alpha\beta}, \quad \text{for all} \quad \alpha, \beta \in G, \tag{4}$$

where $< \mu_{\alpha,\beta} >$ is the automorphisms of $V$ given by conjugation by $\mu_{\alpha,\beta} \in M$. The action

$$\widehat{C}^1(G, \Gamma) \times \widehat{C}^2(G, \Gamma) \stackrel{*}{\to} \widehat{C}^1(G, \Gamma)$$

is given by $(a, b) * (\eta, \mu) = (\eta', \mu')$, where

$$\eta'_\alpha =< a_\alpha > \eta_\alpha \qquad \text{and} \qquad \mu'_{\alpha,\beta} = a_\alpha \eta_\alpha(a_\beta) = \mu_{\alpha,\beta} a^{-1}_{\alpha\beta} b_{\alpha,\beta}$$

for all $\alpha, \beta \in G$.

Two 2-cochains $(\eta, \mu)$ and $(\eta', \mu')$ are said *weakly cohomologous* if there exists a weak 1-cochain $(a, b)$ such that $(a, b) * (\eta, \mu) = (\eta', \mu')$. Hacque defines the set $\overline{H}^2(G, \Gamma)$ of weak 2-cohomology of the group $G$ with coefficients in the group-ring $\Gamma$ as the set of weak cohomology classes of 2-cochains in $\widehat{C}^2(G, \Gamma)$. On the other hand, he defines a (strict and generalized) 2-*cocycle* just as a generalized 2-cochain $(\eta, \mu)$ satisfying the condition

$$\eta_\alpha(\mu_{\beta,\gamma}) \mu_{\alpha,\beta\gamma} \mu^{-1}_{\alpha\beta,\gamma} \mu^{-1}_{\alpha,\beta} = 1 \tag{5}$$

for all $\alpha, \beta, \gamma \in G$, $\tilde{Z}^2(G,\Gamma)$ denotes the set of 2-cocycles of G in $\Gamma$. It happen then that for any strict 1-cochain $(a,0)$ and any 2-cocycle $(\eta,\mu)$ the 2-cochain $(a,0) * (\eta,\mu)$ is again a 2-cocycle and so the above action restricts to another one

$$\tilde{C}^1(G,\Gamma) \times \tilde{Z}^2(G,\Gamma) \xrightarrow{*} \tilde{Z}^2(G,\Gamma),$$

see [11]. Two 2-cocycles $(\eta,\mu)$ and $(\eta',\mu')$ are said cohomologous if there is a strict 1-cochain $(a,0)$ such that $(\eta',\mu') = (a,0) * (\eta,\mu)$ and the set of cohomology classes of 2-cocycles $\tilde{H}^2(G,\Gamma)$ is Hacque's non-abelian cohomology of $G$ with coefficients on the group ring $\Gamma$.

Now, it is clear from the definitions that for any group $G$ and any ring-group $\Gamma$, Hacque's non-abelian cohomology $\tilde{H}^2(G,\Gamma)$ and Dedecker's $\mathbf{H}^2(G,\Phi(\Gamma))$ are the same as sets. So most of the results stated for Dedecker's cohomology could be used for Hacque's one.

Hacque has the project of developing Hochschild-Serre sequences in non-abelian cohomology, using rings-groups as coefficients (see [12]), but since the category $Gp$ of groups is a faithful subcategory of the category $\mathcal{C}$ of rings-groups via the functor $I(-)$, which takes a group $G$ to the ring-group $I(G) = (\mathbb{Z}(G), G)$, we have that $\mathrm{Aut}(I(G)) = \mathrm{Aut}(G)$ and $\Phi I(G) = (\rho : G \to \mathrm{Aut}(G))$ is the canonical crossed module of automorphisms of the group $G$. Now, there are already Hochschild-Serre exact sequences for non-abelian cohomology of groups using Dedecker's $\mathbf{H}^2$ see [2]. In fact, given a short exact sequence of groups (not necessarily abelian groups)

$$\mathcal{S} : 1 \to N \xrightarrow{i} E \xrightarrow{p} G \to 1$$

and a group $H$ on which $G$ acts via a group homomorphism $\varphi : G \to \mathrm{Aut}(H)$), there is a five terms exact sequence of sets with distinguished elements

$$\begin{aligned} 1 \to \mathrm{Der}_\varphi(G,H) \to \mathrm{Der}_{\varphi p}(E,H) \to \mathrm{Hom}_\varphi(\mathcal{S},H) \\ \to \mathbf{H}^2_\varphi(G,\Phi I(H)) \to \mathbf{H}^2_{\varphi p}(E,\Phi I(H)) \end{aligned}$$

where $\mathrm{Der}_\varphi(G,H)$ is the set of $\varphi$-derivations of $G$ by $H$ (pointed by the trivial derivation), $\mathrm{Hom}_\varphi(\mathcal{S},H)$ is the set of crossed module homomorphisms $\gamma = (\gamma_1,\gamma_0)$ from the crossed module $N \xrightarrow{i} E$ (where the action of $E$ on $N$ is given by conjugation) to $\Phi I(H) = (\rho : H \to \mathrm{Aut}(H))$ which make commutative the following diagram

$$\begin{array}{ccccc} N & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & G \\ {\scriptstyle \gamma_1}\downarrow & & {\scriptstyle \gamma_0}\downarrow & & {\scriptstyle q\varphi}\downarrow \\ H & \xrightarrow{\ \rho\ } & \mathrm{Aut}(H) & \xrightarrow{\ q\ } & \mathrm{Aut}_e(H) \end{array}$$

and finally $\mathbf{H}^2_\varphi(G, \Phi I(H))$, which is Dedecker's non-abelian cohomology of $G$ –relative to the create $\varphi$– with coefficients in the crossed module of automorphisms of $H$, is equal to Hacque's $\widetilde{H}^2_\varphi(G, I(H))$. So there is already Hochschild-Serre sequences for non-abelian cohomology with coefficients in those rings-groups of the form $I(G)$.

## 3. Extensions of Rings-Groups by Groups

The cohomology $\widetilde{H}^2(G, \Gamma)$ was introduced by Hacque to classify equivalence classes of extensions of the ring-group $\Gamma$ by the group $G$. An extension of $\Gamma$ by $G$ is a mixed sequence

$$1 \to \Gamma = [V; M] \to \Delta = [U; N] \to G \to 1 \tag{6}$$

in which $V$ is a subring of $U$, $M$ is a normal subgroup of $N$ and the following three conditions are satisfied

(i) The group $G$ is the quotient group of $N = Gr(\Delta)$ by its normal subgroup $M = Gr(\Gamma)$.

(ii) The action of $N$ on $U$, given by conjugation, restricts to an action of $N$ on $V$.

(iii) For any $\alpha \in G$ let $N_\alpha$ be the set of elements in $N$ whose class in the quotient group $G$ is $\alpha$ and let $U_\alpha$ be the sub-$(V-V)$-bimodule of $U$ generated by the elements in $N_\alpha$. Then the $(V-V)$-bimodule $U$ admits a decomposition as a direct sum $U = \bigoplus_{\alpha \in G} U_\alpha$.

Let us note now that, by condition (ii), the canonical group homomorphism $v : N \to \mathrm{Aut}(\Delta)$ (which takes any element $n \in N$ to the automorphisms of $U$ given by conjugation by $n$) induces, by restriction to $V$, an automorphisms of $\Gamma$ and so the extension (6) gives a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & G \longrightarrow 1 \\ & & \| & & \downarrow{\scriptstyle v} & & \\ & & M & \longrightarrow & \mathrm{Aut}(\Gamma) & & \end{array}$$

which represents a non-singular extension of $G$ by the crossed module $\Phi(\Gamma)$ in Dedecker's sense. So any extension of $G$ by $\Gamma$ has associated a non-singular extension of $G$ by the crossed module $\Phi(\Gamma)$.

On the other hand, two extensions $\Delta$ and $\Delta'$ of $G$ by $\Gamma$ are equivalent if there exists a ring-group automorphisms $\varepsilon : \Delta \to \Delta'$ which makes commutative the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \varepsilon} & & \| & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Delta' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where the commutativity of the left hand side implies that the square

$$\begin{array}{ccc} Gr(\Delta) & \longrightarrow & G \\ {\scriptstyle Gr(\varepsilon)}\downarrow & & \| \\ Gr(\Delta') & \longrightarrow & G \end{array}$$

is commutative. Let us denote $\mathbf{Ext}(G, \Gamma)$ the set of equivalence classes of extensions of $G$ by $\Gamma$. It is then clear that two equivalent extensions in Hacque's sense induce equivalent extensions in Dedecker's sense and so we have a well defined map

$$\mathbf{Ext}(G, \Gamma) \to \mathrm{Ext}(G, \Phi(\Gamma)).$$

Finally let us note that the composition

$$\mathbf{Ext}(G, \Gamma) \to \mathrm{Ext}(G, \Phi(\Gamma)) \overset{\varphi}{\cong} \mathbf{H}^2(G, \Phi(\Gamma)) = \widetilde{H}^2(G, \Gamma)$$

is just Hacque's isomorphism $\xi_{G,\Gamma}$, see [11, Theorem 4.4], but let us say that to see that the above map is really a bijection we need Hacque's definition of crossed product of a group by a ring-group.

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