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Stabilization of the Wave Equation with Unbounded Feedback of Finite Range

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1. Introduction and statement of the main result

Let Ω be a bounded connected open domain in R^N with smooth boundary Γ. We consider the wave equation

$$y''(t,x) = \Delta y(t,x) + u(t)g(x), \quad (t,x) \in (0,T] \times \Omega, \tag{1.1}$$

$$y(0,x) = y_0(x), \quad y'(0,x) = y_1(x); \quad x \in \Omega, \tag{1.2}$$

$$y(t,x) = 0, \quad (t,x) \in (0,T] \times \Gamma, \tag{1.3}$$

where $'$ stands for $\frac{\partial}{\partial t}$, $g \in L^2(\Omega)$. g represents the spatial weighting function relative to the control function u. The control u is demanded to be expressed in feedback form acting only on the velocity vector

$$u(t) = F(y'(t,\cdot)). \tag{1.4}$$

Our study will focus on the closed loop feedback system (1.1)-(1.3), (1.4). Qualitatively, this means that under minimal assumptions on g we seek an appropriate feedback operator F so that all the corresponding solutions decay to zero as $t \to \infty$ in the strongest possible norm. On the other hand, the system above is consistent with the fact that practical considerations of feasibility and implementation demand that only finitely many controllers should act upon the system. Some results can be deduced from earlier papers where different techniques are used: controllability of an auxiliary system in [7], LaSalle's principle in [8], multiplier methods in [1]. In order to present the closest result to ours, we introduce the self-adjoint positive operator A defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad Ay = -\Delta y. \tag{1.5}$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be the orthonormal basis (in $L^2(\Omega)$) of eigenfunctions of A, with $\{\lambda_i\}_{i=1}^{\infty}$ as corresponding eigenvalues. From [1] we quote the following result:

THEOREM 1.1. *Assume that the initial conditions satisfy* $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$ *and consider the feedback*

$$u(t) = -\int_\Omega g(x)y'(t,x)\,dx, \tag{1.6}$$

then the solution of the feedback sytem (1.1)-(1.3), (1.6) satisfies

$$\lim_{t\to\infty}\left(\|y(t,\cdot)\|^2_{H_0^1(\Omega)} + \|y'(t,\cdot)\|^2_{L^2(\Omega)}\right) = 0$$

if, and only if

$$\int_\Omega g(x)\psi_i(x)\,dx \neq 0 \quad \forall i. \tag{1.7}$$

Our main result is essentially an improvement of Theorem 1.1 above:

THEOREM 1.2. *Assume that the initial conditions satisfy* $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y_1 \in H_0^1(\Omega)$ *and consider the feedback*

$$u(t) = \int_\Omega g(x)\Delta y'(t,x)\,dx, \tag{1.8}$$

then the solution of the closed loop system (1.1)-(1.3), (1.8) satisfies

$$\lim_{t\to\infty}\left(\|\Delta y(t,\cdot)\|^2_{L^2(\Omega)} + \|y'(t,\cdot)\|^2_{H_0^1(\Omega)}\right) = 0$$

if and only if (1.7) holds.

As mentioned above, Theorem 1.2 improves a stabilization result stated in [1]. The approach adopted by the authors in [1] is based on the background of unbounded sectorial sesquilinear forms developed in [3]. In the present paper, the stabilization occurs via an unbounded feedback and for stronger norms. The proof given below is similar in some steps to that of [1], [4]. However, the main difference between the present paper and these references is that the background mentioned above is dispensed with and does not seem appropriate in our closed loop system.

2. Proof of Theorem 1.2

2.1. Preliminaires and well posedness. Recall that for $\alpha > 0$, $D(A^\alpha)$ can be topologized by the scalar product

$$(z_1, z_2)_{D(A^\alpha)} = \sum_i \lambda_i^{2\alpha}\Big(\int_\Omega z_1\psi_i\,dx\Big)\Big(\int_\Omega z_2\psi_i\,dx\Big).$$

In particular we have $D(A^{\frac{1}{2}}) \equiv H_0^1(\Omega)$ (equivalent norms). For other identifications between $D(A^\alpha)$ and Sobolev spaces we refer to [2], [6]. More generally, for a given Hilbert space Z, the corresponding scalar product and norm will be denoted respectively by $(\ ,\ )_Z$ and $\|\ \|_Z$. Throughout this paper, C will denote a generic constant and any dependence on a parameter, say σ, will be mentioned by C_σ. For $t \geq 0$ fixed, $y(t)$ will denote the function $x \mapsto y(t,x)$.

With control $u(t)$ in the feedback form (1.8) we are led to consider

$$\text{(2.1)} \qquad \begin{cases} y'' + Ay + (g, Ay')_{L^2(\Omega)}g = 0, \\ \\ y(0) = y_0 \in D(A), \quad y'(0) = y_1 \in D(A^{\frac{1}{2}}). \end{cases}$$

Let us introduce the following variational problem which will be used below

$$\text{(2.2)} \qquad \begin{aligned} &(y'', Av)_{L^2(\Omega)} + (Ay, Av)_{L^2(\Omega)} \\ &\qquad + (g, Ay')_{L^2(\Omega)}(g, Av)_{L^2(\Omega)} = 0, \quad \forall v \in D(A). \end{aligned}$$

Then by Galerkin method we can show

Proposition 2.1. *The equation (2.1) has a unique solution such that*

$$y \in C(0,T; D(A)), \quad y' \in C(0,T, D(A^{\frac{1}{2}})).$$

Furthermore, the mapping $S_t : (y_0, y_1) \mapsto (y(t), y'(t))$ defines a strongly continuous semigroup of contraction over $D(A) \times D(A^{\frac{1}{2}})$.

Proof (sketch). The proof is based on techniques that are the same as the ones used in [5], [6]. To show the existence we can use the Galerkin method with the set of eigenfunctions of A as a basis. Let us choose $y_{0m}, y_{1m} \in \operatorname{span}(\psi_1, \cdots, \psi_m)$ such that

$$\text{(2.3)} \qquad y_{0m} \to y_0 \text{ in } D(A), \quad y_{1m} \to y_1 \text{ in } D(A^{\frac{1}{2}}).$$

Let us define $y_m(t)$ as a solution of

$$(2.4)\quad \begin{cases} (y''_m(t), \lambda_i\psi_i)_{L^2(\Omega)} + (Ay_m(t), \lambda_i\psi_i)_{L^2(\Omega)} \\ \qquad\qquad + (Ay'_m(t), g)_{L^2(\Omega)}(g, \lambda_i\psi_i)_{L^2(\Omega)} = 0, \quad 1 \le i \le m, \\ \\ y_m(t) \in \operatorname{span}(\psi_1, \cdots, \psi_m), \quad y_m(0) = y_{0m}, \quad y'_m(0) = y_{1m}. \end{cases}$$

Then for $y_m(t) = \sum_{i=1}^m h_{im}(t)\psi_i$, we multiply (2.4) by h'_{im} to obtain by summation in i:

$$(2.5)\qquad \frac{1}{2}\frac{d}{dt}\left(\|y'_m(t)\|^2_{D(A^{\frac{1}{2}})} + \|y_m(t)\|^2_{D(A)}\right) + (g, Ay'_m)^2_{L^2(\Omega)} = 0.$$

It follows that

$$(2.6)\qquad \|y'_m(t)\|^2_{D(A^{\frac{1}{2}})} + \|y_m(t)\|^2_{D(A)} \le C, \quad \forall m,$$

so that we can extract a subsequence $(y_{m_k}(t))$ such that $y_{m_k} \to v$ weakly in $L^2(0,T;D(A))$, $y'_{m_k} \to w$ weakly in $L^2(0,T;D(A^{\frac{1}{2}}))$ and it is easy to see that $w = v'$. Furthermore, one can show that v is a solution for (2.1). The remaining part of the proof is standard. ∎

Remark 2.1. The set of initial conditions (y_0, y_1) in (2.1) for which we have

$$(2.7)\qquad \|y''(t)\|^2_{D(A^{\frac{1}{2}})} + \|y'(t)\|^2_{D(A)} < \infty, \quad \forall t \ge 0,$$

is dense in $D(A) \times D(A^{\frac{1}{2}})$. Indeed, if A_g denotes the generator of the semigroup defined in Proposition 2.1, then it is obvious that for $(y_0, y_1) \in D(A_g)$, (2.7) holds.

2.2. STABILIZATION. For $y(t)$ solution of (2.1) we introduce

$$(2.8)\qquad E(y,t) = \|Ay(t)\|^2_{L^2(\Omega)} + \left\|A^{\frac{1}{2}}y'(t)\right\|^2_{L^2(\Omega)}.$$

Then it is easy to see that $E(y,t)$ is nonincreasing in t. On the other hand, it follows that our stabilization problem reduces to the equivalence

$$(2.9)\qquad \lim_{t\to\infty} E(y,t) = 0 \iff (1.7) \text{ holds}.$$

Necessity. If condition(1.7) is violated for some eigenvalue λ_{i_0} which is real and nonzero, then $y(t) = e^{j\lambda_{i_0}t}\psi_{i_0}$ $(j = \sqrt{-1})$ is a solution of (2.1) such that $E(y,t) = 2\lambda^2_{i_0} > 0$.

Sufficiency. The proof of sufficiency is based on the following lemma. Some similar estimates have been established for hyperbolic problems in [1], [4].

LEMMA 2.1. *Assume that (1.7) holds, then for each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that for every $\beta > 0$ we have*

$$\int_0^\infty e^{-\beta t}\|y(t)\|^2_{D(A^{\frac{1}{2}})}\,dt \le C_\varepsilon E(y,0) + \varepsilon \int_0^\infty e^{-\beta t}\|y'(t)\|^2_{D(A^{\frac{1}{2}})}\,dt, \tag{2.10}$$

for every solution of (2.1) for which $y(0) \in D(A)$, $y'(0) \in D(A^{\frac{1}{2}})$.

COROLLARY 2.1. *In addition to (1.7), assume that $(y_0, y_1) \in D(A_g)$, then*

$$\begin{aligned}\int_0^\infty e^{-\beta t}E(y,t)\,dt \ &\le C_\varepsilon[E(y,0) + E(y',0)] \\ &\quad + \varepsilon \int_0^\infty e^{-\beta t}\|y''(t)\|^2_{D(A^{\frac{1}{2}})}\,dt.\end{aligned} \tag{2.11}$$

Remark 2.2. It is trivial to establish Lemma 2.1 and its corollary for $\beta \ge \beta_\varepsilon$, $\beta_\varepsilon > 0$ and fixed using the inequalities

$$\|y(t)\|^2_{D(A^{\frac{1}{2}})} \le CE(y,t) \le CE(y,0).$$

The point is that (2.10) and (2.11) continue to hold for $0 < \beta < \beta_\varepsilon$.

Proof of sufficiency. Since $D(A_g)$ is dense in $D(A) \times D(A^{\frac{1}{2}})$ and $E(y,T)$ is continuous in y uniformly on $T \ge 0$, it is sufficient to consider $(y_0, y_1) \in D(A_g)$. On the other hand, since $\frac{d}{dt}E(y,t) \le 0$, $\lim_{t\to\infty} E(y,t)$ exists and for each $T > 0$,

$$(1 - e^{-\beta T})E(y,T) \le \beta \int_0^T e^{-\beta T}E(y,t)\,dt.$$

Thus, from (2.11), one has

$$\lim_{T\to\infty} E(y,T) \le \beta \int_0^\infty e^{-\beta t}E(y,t)\,dt \le \beta C_\varepsilon[E(y,0) + E(y',0)] + \varepsilon E(y',0).$$

Letting β tend to zero we obtain the left-hand side of (2.9).

Proof of Corollary 2.1. From Lemma 2.1,

$$\int_0^\infty e^{-\beta t}\|y'(t)\|^2_{D(A^{\frac{1}{2}})}\,dt \le C_\varepsilon E(y',0) + \varepsilon \int_0^\infty e^{-\beta t}\|y''(t)\|^2_{D(A^{\frac{1}{2}})}\,dt. \tag{2.12}$$

In addition,

$$\int_0^\infty e^{-\beta t}\|y(t)\|^2_{D(A)}\,dt$$

$$\le \int_0^\infty e^{-\beta t}\left[\|y'\|^2_{D(A^{\frac{1}{2}})} - \frac{d}{dt}(A^{\frac{1}{2}}y', A^{\frac{1}{2}}y)_{L^2(\Omega)} - \frac{1}{2}\frac{d}{dt}(g, Ay)^2_{L^2(\Omega)}\right] dt$$

$$\le \int_0^\infty e^{-\beta t}\|y'(t)\|^2_{D(A^{\frac{1}{2}})}\,dt + \left(y'(0), y(0)\right)_{D(A^{\frac{1}{2}})} + \frac{1}{2}\left(g, Ay(0)\right)^2_{L^2(\Omega)}$$

$$-\beta\int_0^\infty e^{-\beta t}\left[\left(y, y'\right)_{D(A^{\frac{1}{2}})} + \frac{1}{2}(g, Ay)^2_{L^2(\Omega)}\right] dt.$$

For $\beta < 1$, the right-hand side is bounded above by

$$C_1\int_0^\infty e^{-\beta t}\left(\|y'\|^2_{D(A^{\frac{1}{2}})} + \|y\|^2_{D(A^{\frac{1}{2}})}\right) dt + C_2\left(\|y(0)\|^2_{D(A)} + \|y'(0)\|^2_{D(A^{\frac{1}{2}})}\right)$$

$$\le C\left[E(y,0) + \int_0^\infty e^{-\beta t}\left(\|y'\|^2_{D(A^{\frac{1}{2}})} + \|y\|^2_{D(A^{\frac{1}{2}})}\right) dt\right].$$

Inequality (2.11) follows from (2.12), the last inequality and Lemma 2.1.

Proof of Lemma 2.1. Let $\phi \in C^\infty(R)$ satisfy $\phi(t) = 0$ for $t \le 0$ and $\phi(t) = 1$ for $t \ge T$ where $T > 0$ is fixed. Let $w = \phi y$, then using a similar variational form to (2.2), it follows

$$(w'', Av)_{L^2(\Omega)} + (Aw, Av)_{L^2(\Omega)} + (Aw', g)_{L^2(\Omega)}(g, Av)_{L^2(\Omega)}$$
$$= (h, Av)_{L^2(\Omega)}, \quad \forall v \in D(A),$$

where

$$h = 2\phi' y' + \phi'' y + \phi'(g, Ay)_{L^2(\Omega)}g.$$

h vanishes for $t \ge T$ so that $\int_0^\infty \|h(t)\|^2_{L^2(\Omega)}dt \le C_T E(y,0)$. The spaces $L^2(\Omega)$, $D(A^{\frac{1}{2}})$, $D(A)$ now refer to the complexifications of the original spaces. The corresponding scalar products are extended in the standard way to the complexifications. Let ω be the complex parameter with $\operatorname{Im}\omega < 0$ and let W denote the Fourier transform of w

$$W(\omega) = \int_0^\infty e^{-j\omega t}w(t)dt, \quad j = \sqrt{-1}.$$

Then W satisfies the variational problem: $W \in D(A)$ and

$$(AW, AV)_{L^2(\Omega)} - \omega^2(W, AV)_{L^2(\Omega)} + j\omega(AW, g)_{L^2(\Omega)}(g, AV)_{L^2(\Omega)}$$
$$= (H, AV)_{L^2(\Omega)}, \quad \forall V \in D(A)$$

or, equivalently

$$AW - \omega^2 W + j\omega(AW, g)_{L^2(\Omega)} g = H, \tag{2.13}$$

where H is the Fourier transform of h. Then we have

LEMMA 2.2. *For each complex ω with $\operatorname{Im}\omega \le 0$ and each $H \in L^2(\Omega)$, the equation (2.13) has a unique solution W such that*

$$\|W\|_{D(A)} \le C_\omega \|H\|_{L^2(\Omega)}. \tag{2.14}$$

Once Lemma 2.2 is established, the proof of Lemma 2.1 can be completed by an argument similar to that used in the proof of Lemma 2 of [4]. Write $\omega = \alpha - j\beta$, $\beta > 0$ and small. Then

$$e^{-\beta t} w(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\alpha t} W(\omega, x)\, d\alpha,$$

$$e^{-\beta t} w'(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j\alpha t} (j\omega) W(\omega, x)\, d\alpha.$$

By Parseval's equality,

$$\int_0^\infty e^{-2\beta t}\left\|A^{\frac12} w\right\|^2_{L^2(\Omega)} dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|A^{\frac12} W(\omega,x)\right\|^2_{L^2(\Omega)} d\alpha,$$

$$\int_0^\infty e^{-2\beta t}\left\|A^{\frac12} w'\right\|^2_{L^2(\Omega)} dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|\omega A^{\frac12} W(\omega,x)\right\|^2_{L^2(\Omega)} d\alpha.$$

Let $\gamma > 0$ be so large that $\gamma^{-2} \le \varepsilon$. For $|\alpha| \le \gamma$ and $|\beta| \le \delta(\gamma)$ we obtain from Lemma 2.2

$$\int_{-\gamma}^{\gamma} \|W\|^2_{D(A^{\frac12})} d\alpha \le C_\gamma \int_{-\infty}^{+\infty} \|H\|^2_{L^2(\Omega)} d\alpha. \tag{2.15}$$

We also have

$$\begin{aligned}\int_{|\alpha|>\gamma} \|W\|^2_{D(A^{\frac12})} d\alpha &\le \int_{|\alpha|>\gamma} \frac{\alpha^2}{\gamma^2}\|W\|^2_{D(A^{\frac12})} d\alpha \\ &\le \varepsilon \int_{-\infty}^{+\infty} \|\omega W\|^2_{D(A^{\frac12})} d\alpha \qquad (2.16)\\ &\le 2\pi\varepsilon \int_0^\infty e^{-2\beta t}\|w'\|^2_{D(A^{\frac12})} dt.\end{aligned}$$

Adding (2.15) and (2.16) gives

$$2\pi \int_0^{\infty} e^{-2\beta t} \|w\|^2_{D(A^{\frac{1}{2}})} dt \le C_\gamma \int_{-\infty}^{+\infty} \|H\|^2_{L^2(\Omega)} d\alpha + 2\pi\varepsilon \int_0^{\infty} e^{-2\beta t} \|w'\|^2_{D(A^{\frac{1}{2}})} dt$$

$$\le C_\varepsilon \int_0^{\infty} e^{-2\beta t} \|h(t)\|^2_{L^2(\Omega)} dt + 2\pi\varepsilon \int_0^{\infty} e^{-2\beta t} \|w'\|^2_{D(A^{\frac{1}{2}})} dt$$

$$\le C_\varepsilon E(y,0) + 2\pi\varepsilon \int_0^{\infty} e^{-2\beta t} \|w'\|^2_{D(A^{\frac{1}{2}})} dt,$$

for some constant C_ε depending on ε. From the definition $w = \phi y$, we have

$$\int_0^{\infty} e^{-2\beta t} \|y(t)\|^2_{D(A^{\frac{1}{2}})} dt \le \int_0^{T} \|y(t)\|^2_{D(A^{\frac{1}{2}})} dt + \int_0^{\infty} e^{-2\beta t} \|w(t)\|^2_{D(A^{\frac{1}{2}})} dt$$

$$\le C_\varepsilon E(y,0) + \varepsilon \int_0^{\infty} e^{-2\beta t} \|w'(t)\|^2_{D(A^{\frac{1}{2}})} dt,$$

$$\int_0^{\infty} e^{-2\beta t} \|w'(t)\|^2_{D(A^{\frac{1}{2}})} dt \le C \left[\int_0^{T} \|y(t)\|^2_{D(A^{\frac{1}{2}})} dt + \int_0^{\infty} e^{-2\beta t} \|y'(t)\|^2_{D(A^{\frac{1}{2}})} dt \right]$$

$$\le C \left[E(y,0) + \int_0^{\infty} e^{-2\beta t} \|y'(t)\|^2_{D(A^{\frac{1}{2}})} dt \right].$$

This completes the proof of Lemma 2.1. ∎

Proof of Lemma 2.2. For a given complex parameter $\omega = \alpha + j\beta$ with $\beta \le 0$, let us consider the following operators from $D(A)$ to $L^2(\Omega)$:

$$A_\omega V = AV - \omega^2 V + j\omega (AV, g)_{L^2(\Omega)} g,$$
$$B_\omega V = AV - \omega^2 V,$$
$$P_\omega V = j\omega (AV, g)_{L^2(\Omega)} g.$$

Since P_ω is of finite dimensional range and B_ω-bounded, P_ω has B_ω-bound zero [3, Problem 1.14, p. 196]. On the one hand, it is easy to see that the compactness of the resolvent of B_ω implies that A_ω has also compact resolvent [3, Theorem 3.17, p. 214]. On the other hand, since A_ω is bounded from $D(A)$ to $L^2(\Omega)$, the proof of the lemma reduces to see that A_ω is bijective. By Fredholm alternative it suffices to show that A_ω is injective. Suppose that $A_\omega W = 0$, then

$$(A_\omega W, AV)_{L^2(\Omega)} = 0, \quad \forall V \in D(A). \tag{2.17}$$

On the other hand, we have

$$\begin{aligned}\mathrm{Re}(A_\omega W, AW)_{L^2(\Omega)} =& \|W\|^2_{D(A)} - \alpha^2 \|W\|^2_{D(A^{\frac{1}{2}})} + \beta^2 \|W\|^2_{D(A^{\frac{1}{2}})} \\ & - \beta \left|(g, AW)_{L^2(\Omega)}\right|^2,\end{aligned}$$

$$\mathrm{Im}(A_\omega W, AW)_{L^2(\Omega)} = -2\alpha\beta \|W\|^2_{D(A^{\frac{1}{2}})} + \alpha \left|(g, AW)_{L^2(\Omega)}\right|^2.$$

If $\alpha = 0$, it follows from $\mathrm{Re}(A_\omega W, AW)_{L^2(\Omega)} = 0$ that $W = 0$. If $\alpha \neq 0$, then $\mathrm{Im}(A_\omega W, AW)_{L^2(\Omega)} = 0$ implies that $\left|(g, AW)_{L^2(\Omega)}\right|^2 - 2\beta\|W\|^2_{D(A^{\frac{1}{2}})} = 0$; thus if $\beta < 0$, $W = 0$ once again, while if $\beta = 0$,

$$\left|(g, AW)_{L^2(\Omega)}\right|^2 = 0. \tag{2.18}$$

Equation (2.17) then yields

$$2(AW, AV)_{L^2(\Omega)} - \omega^2 (W, AV)_{L^2(\Omega)} = 0, \quad \forall V \in D(A). \tag{2.19}$$

Taking into account the hypothesis (1.7), (2.18) and (2.19) can hold simultaneously only if $W = 0$. This completes the proof. ∎

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