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Quasilinear Elliptic Equations with Arbitrary Growth Nonlinearity and Data Measures

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1. Introduction

The purpose of this note is to study existence of weak solutions for the quasilinear elliptic problem with Dirichlet boundary conditions

$$(P_\lambda) \qquad \begin{cases} -u''(t) = j(t, u(t), u'(t)) + \lambda f & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda \in \mathbb{R}$, $j : (0,1) \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ is measurable and continuous with respect to u and u', and f is a given finite nonnegative measure on $]0,1[$. When f is regular, it is proved in [6] that if (P_λ) has a nonnegative supersolution in $W_0^{1,\infty}$ then (P_λ) has a solution in $W_0^{1,\infty} \cap W^{2,p}$. Note that here the supersolution is required to vanish at the boundary. This provides an a priori pointwise estimate for $u'(0)$ and $u'(1)$. The boundedness on u' on the whole set $]0,1[$ is then obtained by a maximum principle applied to the equation satisfied by $|u'|^2$. The convexity of $s \to j(t,r,s)$ is there an essential ingredient.

When f is irregular, one must work with "weak solutions" for which u' is not bounded. As a consequence the techniques usually used to prove existence and based on a priori L^∞–estimate on u and u' fail.

When j does not depend on u but has arbitrary growth with respect to u', existence of weak solution for this type of problems have been obtained in [1] provided we known the existence of a nonnegative weak supersolution in $W_{loc}^{1,\infty}$. When j depends on u' with subquadratic growth and arbitrary growth with respect to u. The same result has been obtained in [2] but in more space dimensions.

In the present paper we are particularly interested in situations where f is irregular and where the growth of j with respect to u and u' is arbitrary. Obviously the classical approach fails to provide existence and new techniques have to be used. We describe some of them here.

In Section 2 we give the precise setting of the problem and state the main result. We present in Section 3 an approximate equation for (P_λ). We prove in Section 4, that the existence of weak supersolutions implies the existence of weak solutions.

2. Statement of the main result

Throughout this paper we suppose

(2.1) f is a nonnegative finite measure on $]0,1[$ and $j : [0,1]\times\mathbb{R}\times\mathbb{R} \to [0,+\infty[$ is such that

(2.2) j is measurable, almost everywhere t, $(r,s) \to j(t,r,s)$ is continuous.

(2.3) j is non decreasing in r and convex in s.

(2.4) $\forall r,s \in \mathbb{R}$, $j(\cdot,r,s)$ is integrable on $]0,1[$.

(2.5) $j(t,r,0) = \min\{j(t,r,s), r \in \mathbb{R}\} = 0$.

We introduce now the notion of weak solution and weak supersolution of the problem (P_λ) used here.

Definition 2.1. A function u is said to be a weak solution of the problem (P_λ), if

$$\begin{cases} u \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -u'' = j(\cdot,u,u') + \lambda f \quad \text{in} \quad D'(0,1). \end{cases} \tag{2.6}$$

Definition 2.2. A weak supersolution of the problem (P_λ) is a function w such that

$$\begin{cases} w \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -w'' \geq j(\cdot,w,w') + \lambda f \quad \text{in} \quad D'(0,1). \end{cases} \tag{2.7}$$

Remark 2.1. In (2.6) and (2.7) $u, w \in W^{1,\infty}_{loc}(0,1)$, using then (2.4) we have $j(\cdot,u,u') \in L^1_{loc}(0,1)$ (resp. $j(\cdot,w,w') \in L^1_{loc}(0,1)$). Hence every term in (2.6) and (2.7) makes sense.

This enables us to state the main result of this paper.

THEOREM 2.1. *Assume that (2.1)-(2.5) hold. Assume that there exists a weak supersolution w of the problem*

$$(2.8)\qquad \begin{cases} u \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -u'' = j(\cdot, u, u') + f \quad \text{in} \quad D'(0,1). \end{cases}$$

Then there exists a weak solution u of (P_λ) for all $0 \le \lambda \le 1$, and

$$0 \le u \le w \quad \text{a.e. in} \quad [0,1].$$

Remark 2.2. 1) It should be noted that there is not growth restriction on the "lower order nonlinearity" of j as a function in u'. Hence the present theorem extends some results in [2].

2) It is possible to consider various extensions of problem (P_λ), for exemple to other boundary conditions or to a problem with periodicity conditions.

3) For a given problem, on can try several methods to exhibit a supersolution. Of cours there is no methodology, but, usually, one should try functions which are locally "simple" (constants, linear, eigenfunctions of simple operators). The existence of such a supersolution is actually a very general fact, as proved in [3].

First we introduced regularized versions of (2.8) with smaller j_n with linear growth. One advantage of this approach is that the corresponding solutions u_n, are such that $u_n \in W^{1,\infty}_0$ and $0 \le u_n \le u_{n+1} \le w$.

3. AN APPROXIMATE EQUATION

In this section, we define an "approximate" equation of (P_λ). For $n \ge 1$, we consider the approximation $\widehat{j_n}(t,r,\cdot)$ of $j(t,r,\cdot)$ defined by

$$(3.1)\qquad \widehat{j_n}(t,r,s) := \begin{cases} j(t,r,-n) + j'_s(t,r,-n)(s+n) & \text{if} \quad s \le -n \\ j(t,r,s) & \text{if} \quad |s| < n \\ j(t,r,n) + j'_s(t,r,n)(s-n) & \text{if} \quad s \ge n, \end{cases}$$

where j'_s denotes a section of the sub differential of j with respect to s. Then we set

$$(3.2)\qquad j_n(t,r,s) = \widehat{j_n}(t,r,s) 1_{[w \le n]}$$

where w is a supersolution of (P_λ). Then j_n satisfies (2.2)-(2.5) and,

$$j_n \le j1_{[w\le n]}, \quad j_n \le j_{n+1}. \tag{3.3}$$

According to the result in [1], there exists a sequence (u_n) of solutions of the problem

$$(P_{\lambda,n}) \qquad \begin{cases} u_{n+1} \in W_0^{1,\infty}(0,1) \\ -u''_{n+1} = j_{n+1}(\cdot, u_n, u'_{n+1}) + \lambda f_{n+1} \quad \text{in } \; D'(0,1) \end{cases}$$

where $u_1 = \lambda G(f_1) \ge 0$ and $f_n = f1_{[w\le n]}$.

4. Estimates. Passing to the limit

We start this section with the following three lemmas, (see [1]).

LEMMA 4.1. *Let $a(t) \in L^1_{loc}(0,1)$, $v \in W^{1,1}_{loc}(0,1) \cap C_0[0,1]$ such that*

$$\begin{cases} a(t)v'(t) \in L^1_{loc}(0,1) \\ -v'' - av' \ge 0 \quad \textit{in } \; D'(0,1). \end{cases} \tag{4.1}$$

Then $v \ge 0$ in $[0,1]$.

LEMMA 4.2. *Let $u \in W_0^{1,1}(0,1)$, $v \in L^\infty(0,1)$ such that*

$$0 \le u \le v \quad \textit{in } \; (0,1), \quad 0 \le -u' \quad \textit{in } \; D'(0,1). \tag{4.2}$$

Then,

$$u \in W^{1,\infty}_{loc}(0,1) \cap W_0^{1,1}(0,1)$$

and

$$|u'(x)| \le \frac{1}{d(x;a,b)}\Big(c(a,b) + \|v\|_{L^\infty(0,1)}\Big) \tag{4.3}$$

for all $0 < a < b < 1$. Where $d(x;a,b) = \min(b-x, x-a)$ and $c(a,b)$ is a constant.

Remark 4.1. Lemma 4.2 will provide $W^{1,\infty}_{loc}$ estimates for the approximate solution u_n. But this estimate do not allow us to pass to the limit in the nonlinear terms. We need the strong convergence of u_n in $W^{1,\infty}_{loc}(0,1)$. We obtain this result from the following lemma.

LEMMA 4.3. *Let* $(u_n) \subset W_0^{1,\infty}(0,1)$, *such that*

$$u_n \text{ converging strongly in } L^\infty(0,1) \text{ to } u \tag{4.4}$$

$$u_n' \text{ converging strongly in } L^1_{loc}(0,1) \text{ to } u' \text{ and a.e. in }]0,1[\tag{4.5}$$

$$0 \le u_n \le u, \quad -u_n'' \ge 0 \quad \text{in } D'(0,1). \tag{4.6}$$

Then

$$u_n' \text{ converges to } u' \text{ strongly in } L^\infty_{loc}(0,1).$$

Proof of Theorem 2.1. Let us prove by induction that

$$0 \le u_n \le \min(w,n) = w_n, \quad \text{for all } n \ge 1. \tag{4.7}$$

First from $(P_{\lambda,n})$ we have for all $n \ge 1$

$$\begin{cases} u_{n+1} \in W_0^{1,\infty}(0,1) \\ -u_{n+1}'' \ge 0 \quad \text{in } D'(0,1). \end{cases} \tag{4.8}$$

By Lemma 4.1 we see that

$$u_{n+1} \ge 0 \quad \text{in } [0,1]. \tag{4.9}$$

We easily deduce from the definition of w that

$$-w_n'' \ge j(\cdot, w_n, w_n')1_{[w\le n]} + f1_{[w\le n]} \quad \text{in } D'(0,1). \tag{4.10}$$

For $n = 1$, using (4.8), (4.10) we get

$$\begin{cases} w_1 - u_1 \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -(w_1 - u_1)'' \ge 0 \quad \text{in } D'(0,1) \end{cases} \tag{4.11}$$

hence by Lemma 4.1 we have, $0 \le u_1 \le w_1$. Let us assume $0 \le u_n \le w_n$, then from (4.10), the monotonicity of j in r and (3.3), we have

$$\begin{cases} w_{n+1} \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -w_{n+1}'' \ge j_{n+1}(\cdot, u_n, w_{n+1}') + f_{n+1} \quad \text{in } D'(0,1) \end{cases}$$

hence w_{n+1} is a supersolution of the problem $(P_{\lambda,n})$. Since

$$-(w_{n+1}-u_{n+1})'' \geq j_{n+1}(\cdot, u_n, w'_{n+1})-j_{n+1}(\cdot, u_n, u'_{n+1})+(1-\lambda)f_{n+1} \quad \text{in } D'(0,1)$$

$$-(w_{n+1} - u_{n+1})'' \geq \partial j_{n+1}(\cdot, u_n, u'_{n+1})(w_{n+1} - u_{n+1})' \quad \text{in } D'(0,1)$$

then we have

$$(4.12) \qquad \begin{cases} \theta_{n+1} \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -\theta''_{n+1} - a_n\theta'_{n+1} \geq 0 \quad \text{in } D'(0,1) \\ a_n\theta'_{n+1} \in L^1_{loc}(0,1) \end{cases}$$

where $\theta_{n+1} = w_{n+1} - u_{n+1}$, $a_n \in \partial j_{n+1}(\cdot, u_n, u'_{n+1}) \in L^1_{loc}(0,1)$.

Now, Lemma 4.1 can be applied with the functions a_n and θ_{n+1}. Hence $0 \leq u_{n+1} \leq w_{n+1}$ in $[0,1]$ which proves (4.7) by induction.

By Lemma 4.2 and (4.10), u_n is bounded in $W^{1,\infty}_{loc}(0,1) \cap C_0[0,1]$ independently of n. Therefore, there exists a subsequence, still denoted by (u_n) for simplicity, such that if n tens to ∞ then u_n converges to u strongly in $L^\infty(0,1)$ and u'_{n+1} converges to u' strongly in $L^1_{loc}(0,1)$ and almost everywhere in $]0,1[$. From Lemma 4.3 we then have u'_{n+1} converges to u' strongly in $L^\infty_{loc}(0,1)$, and

$$(4.13) \qquad \|u'_n\|_{L^\infty(a,b)} \leq K(a,b)\Big(c(a,b) + \|w\|_{L^\infty(0,1)}\Big) = c'(a,b).$$

Since $j(t,\cdot,\cdot)$ is continuous in the two last arguments we have for all $0 < a < b < 1$

$$(4.14) \quad j_{n+1}(t, u_n(t), u'_{n+1}(t)) \text{ converges to } j(t, u(t), u'(t)) \text{ a.e. } t \in]0,1[.$$

On the other hand, for a.e. $t \in]0,1[$

$$|j_{n+1}(t, u_n(t), u'_{n+1}(t))| \leq j(t, u(t), u'_{n+1}(t)) \leq \max_{\substack{|r| \leq \|u\|_{L^\infty(a,b)} \\ |s| \leq c'(a,b)}} |j(t,r,s)| = \theta(t)$$

and $\theta \in L^1_{loc}(0,1)$ from (2.4).

Using Lebesgue's dominate convergence Theorem (see [5]), we also have

$$(4.15) \qquad j_{n+1}(\cdot, u_n, u'_{n+1}) \text{ converges to } j(\cdot, u, u') \text{ strongly in } L^1(a,b).$$

Now, we can pass to the limit in $(P_{\lambda,n})$, and if $\varphi \in D(0,1)$ with $\operatorname{supp}\varphi \subset [a,b]$ then

$$0 = \lim_{n\to\infty} < -u''_{n+1} - j_{n+1}(\cdot, u_n, u'_{n+1}) - \lambda f_{n+1}, \varphi > = < -u'' - j(\cdot, u, u') - \lambda f, \varphi > .$$

Where $< \cdot, \cdot >$ denotes the duality pairing between $D'(0,1)$ and $D(0,1)$. This completes the proof. ▮

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