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Mathematical Theory of Musical Scales

M.J. Garmendia Rodríguez and J.A. Navarro González

Instituto "Bárbara de Braganza", 06071 Badajoz, Spain
Dpto. de Matemáticas, Universidad de Extremadura, 06071 Badajoz, Spain

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Our aim is to look for precise definitions of musical concepts. In this work we present the concepts we have been able to derive from the concept of *pitch* (high–low aspect of musical sounds). Now, pitches being the primitive concept, they will be not defined from a previous concept, but from their mutual relationships. Our first task is to elucidate the structure of pitches. Each pitch t is fully determined by a vibrational frequency, but most men are not able to identify the absolute frequency of a pitch; only some ratios of frequencies are clearly identified when two notes are played at a time. Given a pitch t and a real number λ, we shall denote by λt the pitch such that λ is just the ratio between the frequencies corresponding to λt and t. For example, if t corresponds to a frequency of $440Hz$, the $3t$ corresponds to $1320Hz$ and $t/2$ to $220Hz$. So we obtain a free and transitive action of the multiplicative group $\mathcal{I}$ of all positive real numbers on the set of pitches. Moreover, most men sharply identify any pitch t with $2t$, and it is said that $2t$ is one octave higher than t. This is all the structure of pitches we need:

Definition. The *System of Pitches* is a set $\mathcal{P}$, a fixed positive real number $\neq 1$ (which always will be assumed to be 2) and a free transitive action of the group $\mathcal{I}$ on $\mathcal{P}$ (that is to say, a map $f\colon \mathcal{I}\times\mathcal{P}\to\mathcal{P}$, $(\lambda,t)\mapsto\lambda t$, such that $1t=t$, $\lambda(\mu t)=(\lambda\mu)t$ and, for any two $t,t'\in\mathcal{P}$, there exists a unique $\lambda\in\mathcal{I}$ such that $t'=\lambda t$). The elements of $\mathcal{P}$ will be said to be *pitches* and the elements of the group $\mathcal{I}$ will be said to be *intervals*.

The subgroup $2^{\mathbb{Z}}$ defines an equivalence relation on $\mathcal{P}$: $t'\equiv t \Leftrightarrow t'=2^r t$ for some integer number r. The quotient set $\mathcal{O}=\mathcal{P}/\equiv$ is said to be the *Octave*. The group $\mathcal{I}$ acts transitively on $\mathcal{O}$.

Let $\pi\colon \mathcal{P} \to \mathcal{O}$ be the canonical[1] projection. The image $\pi(t)$ of any pitch t in the Octave will be denoted with the corresponding capital letter: $\pi(t) = T$.

Some intervals deserve a proper name: 2 is the *octave*, 3/2 is the *perfect fifth* and 5/4 is the *major third.*

Any magnitude (length, volume, mass,...) is endowed with a free transitive action of the multiplicative group $\mathbb{R}_+$, but in these cases we are not able to identify a preferred ratio except for 1:1. One may wonder about ligth, since it is also determined by a frequency, but the visible spectrum does not contain frequencies in a ratio 2:1.

GEOMETRIC REPRESENTATION OF THE OCTAVE. To look for a geometric representation of some structure means to look for a geometric figure whose geometric structure includes the structure under consideration. The geometric representation of a structure, even if it is not necessary for the validity of any statement, is a valuable guide for our comprehension, due to our penetrating intuition of any kind of geometric structures.

Since $\log_2\colon \mathbb{R}_+ \to \mathbb{R}$ is an isomorphism, the multiplicative group $\mathbb{R}_+$ may be replaced by the aditive group $\mathbb{R}$ in the above definition. Now, once we fix a unit of length and an orientation in a straight line, the group $\mathbb{R}$ clearly acts on such line by traslations: any oriented straight line with a fixed unit of length is a system of pitches. Unfortunately, we may not state that oriented straight lines are systems of pitches, because they have not a distingued segment. In this case the Octave is obtained identifying points determining an integer multiple of the fixed unit of length. Therefore the Octave is an oriented circle with a fixed unit of length; but any circle determines a natural unit: its own length. We see that the wole structure of the Octave coincides exactly with the structure of an oriented circle: the study of the Octave is fully equivalent to the study of the intrinsic geometry of an oriented circle.

We denote pitches by lowercase letters and their projection on the Octave $\mathcal{O}$ by the corresponding capital letter. Let us fix a pitch t and a point of an oriented circle. Given a pitch $t' = \lambda t$, we shall represent $T' = \lambda T$ by the point of the oriented circle determining with the fixed point an arc of $\log_2 \lambda$ turns ($= 360 \cdot \log_2 \lambda$ degrees, that we draw in the counterclockwise sense).

DEFINITION. Two finite non-empty subsets S and S' of the Octave $\mathcal{O}$ are said to be equivalent when $S' = \lambda S$ for some interval λ. Equivalence classes

[1] The adjective "canonical" means that this map is universal, independent of arbitrary choices and cultural traditions; briefly, that it emerges directly from the structure under consideration (the structure of pitches in our case).

of finite subsets of $\mathcal{O}$ are said to be *musical scales*. The number of notes of a scale $\mathbf{S}$ is defined to be the common cardinal number of all finite subsets of $\mathcal{O}$ representing $\mathbf{S}$.

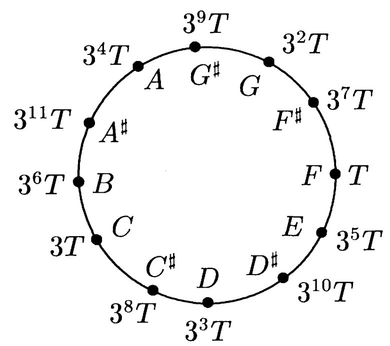
The study of musical scales is equivalent to the study of polygons inscribed in an oriented circle, up to rotations.

SCALES OF FIFTHS. Let $\mathbf{S}$ be the scale defined by a finite subset S of the Octave. From the point of view of melody, one should like that $3T'$, $T'/3 \in S$ whenever $T' \in S$; but this condition contradicts the finiteness of the number of notes, since no power of 3 is a power of 2.

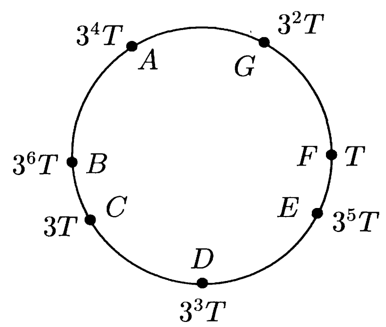
The *scale of fifths* $\mathbf{Q}_n$ of n notes is defined to be the scale represented by $\{T, 3T, 3^2T, \ldots, 3^{n-1}T\}$, where T corresponds to any pitch t, since the scale does not depend on T. This scale has n notes and it clearly has the perfect fifth or dominant $3T'$ of any note T' in the scale, except for $T' = 3^{n-1}T$, as well as the subdominant $T'/3$ of any note T', except for $T' = T$: scales of fifths are the most perfect scales for melodies. Some scales of fifths have a proper name: the *pentatonic* scale when $n = 5$, the *diatonic* scale when $n = 7$ and the *chromatic* scale when $n = 12$. According to the *Britannica*, the pentatonic scale is used more widely than any other scale, and Western art music is one of the few traditions in which pentatonic scales do not predominate.

Here you have the geometrical representation of the chromatic scale and the traditional names of its notes (in general, a sharpped note $\bar{T}^\sharp$ denotes $3^7\bar{T}$ and a flatted note $\bar{T}^\flat$ denotes $3^{-7}\bar{T}$):

Chromatic Scale



Diatonic Scale

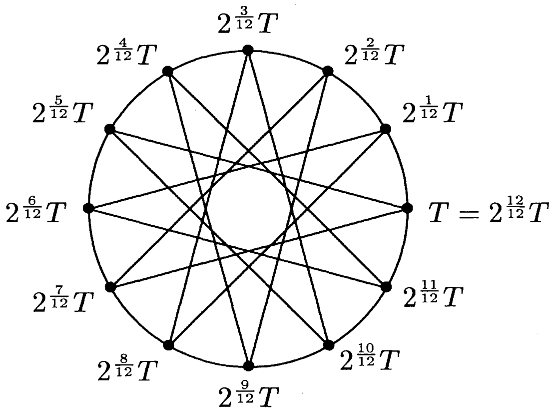


$$T = F = Fa\,, \quad 3T = C = Do\,, \quad 3^2T = G = Sol\,, \quad 3^3T = D = Re\,,$$
$$3^4T = A = La\,, \quad 3^5T = E = Mi\,, \quad 3^6T = B = Si\,, \quad 3^7T = F^\sharp = Fa^\sharp,$$
$$3^8T = C^\sharp = Do^\sharp\,, \quad 3^9T = G^\sharp = Sol^\sharp, \quad 3^{10}T = D^\sharp = Re^\sharp\,, \quad 3^{11}T = A^\sharp = La^\sharp.$$

In the chromatic scale, $A^\sharp = La^\sharp$ is the unique note without perfect fifth. The distance from $A^\sharp = La^\sharp$ to the closest note in the scale, in fact $F = Fa$, is about 0.0196, because $\log_2 3^{12} \approx 19.0196$.

Given a scale $\mathbf{S}$ represented by a subset S of the Octave, the approximation index of perfect fifts $\alpha_3(\mathbf{S})$ is defined to be the infimum of all real numbers α such that the distance from $3T'$ to S is $\leq \alpha$ for any $T' \in S$. Hence, the perfect fifth of any note in the scale may be replaced by a note in the scale at a distance bounded by $\alpha_3(\mathbf{S})$. In the case of the scale of n fifths $\mathbf{Q}_n$, the index $\alpha_3(\mathbf{Q}_n)$ is just the distance from $3^n T$ to T (the distance from $n \log_2 3$ to the closest integer number).

TEMPERED SCALES. From the point of view of modulation, one looks for scales $\mathbf{S}$ such that $T, \lambda T \in S \Rightarrow \lambda T' \in S$ for any other $T' \in S$. That is to say, for scales dividing the Octave in equal parts: scales corresponding to regular polygons. A scale is said to be *tempered* if it divides the Octave into equal parts. Therefore, the tempered scale of n notes $\mathbf{T}_n$is represented by $\{T, rT, r^2T, \dots r^{n-1}T\}$, $r = 2^{1/n}$, where T corresponds to any pitch t. The keys of a well–tunned piano define the tempered scale of 12 notes (the straight lines join each note T' to the best approximations of its dominant $3T'$ and subdominant $T'/3$):



Scales of fifths $\mathbf{Q}_n$ are not tempered, since no power of 3 may be a power of 2. Hence, at least two distances between consecutive notes in $\mathbf{Q}_n$ are not equal. However, even if there are only two different distances α_1, α_2 between consecutive notes, $\mathbf{Q}_n$ may be highly non–tempered, because it may be that $\alpha_2 \gg \alpha_1$.

THEOREM. *The following conditions are equivalent:*

(1) *The scale of n fifths improves the approximation of perfect fifths of any previous scale of fifths:*

$$\alpha_3(\mathbf{Q}_n) < \alpha_3(\mathbf{Q}_m) \quad \textit{for any } m < n.$$

(2) *The scale of n fifths has good temperament. Precisely, there are only two different intervals between consecutive notes in $\mathbf{Q}_n$ and their lengths α_1, α_2 are similar:*

$$\alpha_1 < \alpha_2 < 2\alpha_1$$

(3) *The tempered scale of n notes improves the approximation of perfect fifths of any previous tempered scale more than it increases the number of notes:*

$$n\alpha_3(\mathbf{T}_n) < m\alpha_3(\mathbf{T}_m) \quad \textit{for any } m < n.$$

The numbers satisfying these equivalent conditions are $n = 2, 5, 12, 41, 53, 665 \ldots$.

When the introduction of modulations lead musicians to enlarge the diatonic scale $\mathbf{Q}_7$ so as to obtain a scale of fifths with better temperament, it was not by a coincidence or a cultural fashion that they arrived to the chromatic scale $\mathbf{Q}_{12}$, but a necessary fact unless they would return to the pentatonic scale or would be disposed to accept a scale with 41 notes. Moreover, when the massive use of modulations urged the introduction of tempered scales (with good approximation of perfect fifths, of course), it was not a coincidence that the adopted scale also had 12 notes, hence only requiring a new tunning of musical instruments, not the modification of the number of keys.

But, what about arbitrary scales? is there some better scale? From the point of view of melody, one looks for a scale $\mathbf{S}$ with a good approximation index $\alpha_3(\mathbf{S})$ of perfect fifths. From the point of view of modulations, one asks $\mathbf{S}$ to have a good temperament. From the point of view of harmony one asks that the major chord T', $3T'$, $5T'$ based on any note $T' \in \mathbf{S}$ should have a good approximation in $\mathbf{S}$; that is to say, scales with a good approximation index $\alpha_5(\mathbf{S})$ of major thirds $5T'$, $T' \in \mathbf{S}$. Since we have a sharper perception of small deviations in perfect fifths than in major thirds, we introduce the index $\alpha(\mathbf{S}) = \max\{6\alpha_3(\mathbf{S}), \alpha_5(\mathbf{S})\}$ as a measure of the adaptation of the scale $\mathbf{S}$ to the melodical and harmonical requirements, neglecting modulations by the moment (where tempered scales are the perfect ones). We prove that our usual tempered scale $\mathbf{T}_{12}$ is the best scale with less than 16 notes in a very precise sense:

THEOREM. *Let* $\mathbf{S}$ *be a scale of* n *notes.*

If the index $\alpha(\mathbf{S})$ *is smaller than the index* $\alpha(\mathbf{T}_{12})$ *of the tempered scale of 12 notes, then* $n \geq 16$.

If $n \leq 12$ *and* $\alpha(\mathbf{S}) = \alpha(\mathbf{T}_{12})$, *then* $n=12$, $\alpha_5(\mathbf{S}) = \alpha_5(\mathbf{T}_{12})$ *and* $\alpha_3(\mathbf{S}) \geq \alpha_3(\mathbf{T}_{12})$, *the last inequality being strict unless* $\mathbf{S} = \mathbf{T}_{12}$.

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