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On the Continuability of Solutions of Bidimensional Systems

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1. Introduction

Consider the system:

$$(1) \qquad \begin{aligned} x' &= \alpha(y) - \beta(y)F(x), \\ y' &= -a(t)g(x), \end{aligned}$$

where α, β, f $(F(x) = \int_0^x f(s)ds)$ are real valued continuous functions on $\mathbb{R}$. Moreover a is continuous on $[0, +\infty)$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous such that $xg(x) > 0$ for $x > 0$. Dots denote the differentiation with respect to t. Let us note that in the case $\alpha(y) = y$, $\beta(y) \equiv 1$ system (1) reduces to the equation of the second order:

$$(2) \qquad x'' + f(x)x' + a(t)g(x) = 0.$$

If $a \equiv 1$ and $f \equiv h'$ then (2) is the well known Liénard's equation:

$$x'' + h'(x)x' + g(x) = 0.$$

It is known that in the case that a is sufficiently smooth positive, all solutions of (2) can be extended for all $t > 0$ (see [5]). The case of negative a is quite different. If a is negative at some point then the equation (2) without damping, i.e.

$$x'' + a(t)g(x) = 0,$$

has solutions which are not continuable for all $t > 0$. Moreover for a as above nothing is known about continuability of solutions of (2). The aim of this short note is to study the continuability of solutions of system (1) in the case of a negative in one point.

Let us note that results presented in this note complete earlier papers ([2-4]).

2. Continuability of solutions of (2)

Let $G(x) = \int_0^x g(s)\,ds$. Now we state our first result.

THEOREM 1. *Suppose $a(t_1) < 0$ for some $t_1 > 0$ and $F(x) \le N$ for $x \ge 0$. If the following condition is fullfilled:*

$$\int_0^{+\infty} (1 + G(x))^{-\frac{1}{2}} dx < \infty, \tag{3}$$

then (2) has solution $x(t)$ which is not continuable to $+\infty$.

THEOREM 2. *Let $a(t)$ be continuous satisfying $a(t) < 0$ on an interval $t_1 \le t \le t_2$ with $a(t_2) \le 0$ and there exists $N > 0$ such that $0 \le F(x) \le N$ for $x \ge 0$. Then (2) has a solution $(x(t), y(t))$ defined for $t = t_1$ satisfying:*

$$\lim_{t \to T-} |x(t)| = +\infty, \tag{4}$$

for some $T \in (t_1, t_2]$ if and only if (3) holds

The proofs of Theorems 1 and 2 are based on ideas of proofs presented in [1]. In proofs of [1] it is considered the system

$$x' = y', \quad y' = -a(t)g(x).$$

To prove Theorems 1 and 2 we consider, instead of the above system the following one:

$$x' = y - F(x), \quad y' = -a(t)g(x).$$

Modifying ideas of [1] we obtain the expected results.

Remark 1. The case $x \le 0$ can be proved in a similar way, using the condition $-N \le F(x) \le 0$ for $x \le 0$ and

$$\int_0^{-\infty} (1 + G(x))^{-\frac{1}{2}} dx > -\infty, \tag{5}$$

thus a similar argument may be given in quadrant III of the Phase Plane.

3. The general system (1)

To consider system (1) we assume additionaly: a) α is strictly increasing and such that $y\alpha(y) > 0$ if $y \neq 0$, b) $\beta(y) > 0$, c) $xF(x) > 0$ if $x \neq 0$, d) g is a increasing function on $(-\infty, +\infty)$. From the Theorem 1 we obtain:

THEOREM 3. *Let $a(t_1) < 0$ for some $t_1 > 0$. Suppose there exists $\delta_1 > 0$ and $K > 0$ such that for $t_1 \leq t \leq t_1 + \delta$ either:*

$$y - F(x) \leq \alpha(y) - \beta(y)F(x) \quad \text{for } x > K \text{ and } y > K, \tag{6}$$

or

$$y - F(x) \geq \alpha(y) - \beta(y)F(x) \quad \text{for } x < -K \text{ and } y < -K. \tag{7}$$

If (6) and (3) hold or (7) and (5) hold, then (1) has solutions $(x(t), y(t))$ satisfying $|x(t)| \to \infty$ for some $T_1 > t_1$.

Remark 2. The formulation of the comparison result corresponding to Theorem 2 is easy and we leave it to the reader.

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