EXTRACTA MATHEMATICAE Vol. **11**, Núm. 2, 276 – 281 (1996)

Weakly Complete Semimetrizable Spaces and Complete Metrizability

SALVADOR ROMAGUERA AND SAM D. SHORE*

Escuela de Caminos, Dpto. de Matemática Aplicada, Universidad Politécnica, 46071- Valencia, Spain

Department of Mathematics, University of New Hampshire, Durham, NH- 03824- *USA*

(Presented by F. Montalvo)

AMS *Subject Class.* (1991): 54E25, 54E50 *Received May* 30, 1995

Throughout this paper all topological spaces are T_1 and paracompact spaces are assumed to be regular. Terms and undefined concepts may be found in [7] and [8].

In [4] J. Ceder proved that every paracompact strongly complete semimetrizable space is completely metrizable. This result cannot be generalized to paracompact weakly complete semimetrizable spaces as a known example of L.F. McAuley shows (see [11, Theorem 3.2]). It then arises, in a natural way, the question of obtaining conditions for the complete metrizability of a paracompact weakly complete semimetrizable space. In this note we give an answer to this question. We show that every regular θ, weakly complete asymmetrizable space is a complete Aronszajn space and deduce, among other results, that every paracompact θ, weakly complete semimetrizable space is completely metrizable. Some examples related to the obtained results are also given.

An asymmetric [16] (or an o-metric [12]) on a set X is a nonnegative real-valued function d on $X \times X$ such that $d(x,y) = 0 \iff x = y$.

Consider the following conditions on the asymmetric d:

(i) for all $x, y \in X$, $d(x,y) = d(y,x)$.

(ii) when $d(x,x_n) \to 0$, $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$, then $d(x,y_n) \to 0$.

(iii) when $d(x,x_n) \to 0$ and $d(x_n,y_n) \to 0$, then $d(x,y_n) \to 0$.

*The first author thanks the support of the Conselleria de Educació i Ciencia, Generalitat Valenciana, Grant GV-2223/94.

(iv) for all $x, y, z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$.

Then, d is said to be a symmetric on X ([13],[1]) if it satisfies (i); a θ-metric on X if it satisfies (ii); a γ-metric on X [10] if it satisfies (iii) and a quasi-metric on X [17] if it satisfies (iv). Note that each quasi-metric is a γ-metric and that each γ-metric is a θ-metric.

For any asymmetric d on X, if $S_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ denotes the sphere centered at x of radius $\epsilon > 0$, then:

$$T(d) = \{A \subseteq X : \text{ when } x \in A \text{ then } S_d(x,\epsilon) \subseteq A \text{ for some } \epsilon > 0\}$$

is a topology on X which makes the topological space $(X, T(d))$ a T_1-space. In general $S_d(x,\epsilon)$ need not be a neighborhood of x in $(X, T(d))$. However, if d is a γ-metric, then for each $x \in X$, $\{S_d(x,\epsilon) : \epsilon > 0\}$ is a neighborhood base at x with respect to $T(d)$.

A topological space (X,T) is called asymmetrizable [16], (or o-metrizable [12]) if there is an asymmetric d on X such that $T = T(d)$; in this case we say that d is compatible with T. The notions of a symmetrizable space, of a θ-metrizable space, of a γ-metrizable space and of a quasi-metrizable space are defined in the obvious manner.

A topological space (X,T) is called a θ-space [9] if there is a function $g : \mathbb{N} \times X \longrightarrow T$ such that for each $x \in X$, $x \in \cap\{g(n,x) : n \in \mathbb{N}\}$ and such that when $x_n \in g(n,x)$, $x \in g(n,x_n)$ and $y_n \in g(n,x_n)$ for all $n \in \mathbb{N}$, then x is a cluster point of the sequence $< y_n >$.

Clearly, every θ-space is θ-metrizable. However [16, Example 5.2], provides an example of a θ-metrizable which fails to be a θ-space.

Extending, in a natural way, the usual notions of completeness in symmetrizable and semimetrizable spaces (see [11], [8]) we introduce the notions of a weakly complete asymmetrizable space, of a strongly complete asymmetrizable space and of a Cauchy complete asymmetrizable space. For instance, a space (X,T) will be called weakly complete asymmetrizable if it has a compatible asymmetric d such that every decreasing sequence $< F_n >$ of nonempty closed subsets with $F_n \subseteq S_d(x_n, 2^{-n})$ for some $x_n \in F_n$, has nonempty intersection. Then d is said to be a weakly complete asymmetric for (X,T).

Finally, we also need the following concept:

A regular space (X,T) is called complete Aronszajn [18] if there is a sequence $< \mathcal{B}_n >$ of bases for T such that if $< B_n >$ is a decreasing sequence of subsets of X with $B_n \in \mathcal{B}_n$ for all $n \in \mathbb{N}$, then $\{\text{cl}\, B_n : n \in \mathbb{N}\}$ is a convergent filter base.

PROPOSITION 1. *Each regular θ, weakly complete asymmetrizable space (X, T) is a complete Aronszajn space.*

Proof. Let d be a weakly complete asymmetric for (X, T) and let q be a θ-metric on X compatible with T. Since (X, T) is a Hausdorff first countable space, $x \in \operatorname{int} S_d(x, \epsilon) \cap \operatorname{int} S_q(x, \epsilon)$ for all $x \in X$ and all $\epsilon > 0$. Hence, for each $x \in X$, there is a strictly decreasing sequence $< r_n(x) >$ of positive real numbers such that for each $n \in \mathbb{N}$, $r_n(x) < 2^{-n}$ and $\operatorname{cl} S_q(x, r_n(x)) \subseteq S_d(x, 2^{-n}) \cap S_q(x, 2^{-n})$.

Let for each $n \in \mathbb{N}$, $\mathcal{B}_n = \{\operatorname{int} S_q(x, r_n(x)) : x \in X, k \geq n\}$. Then $\mathcal{B}_n$ is, clearly, a base for T. Now suppose that $< B_n >$ is a sequence of subsets of X such that $B_{n+1} \subseteq B_n$ and $B_n \in \mathcal{B}_n$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $B_n = \operatorname{int} S_q(x_n, r_{k(n)}(x_n))$ for some $x_n \in X$, with $k(n) \geq n$ and $r_{k(n)}(x_n) < 2^{-k(n)}$. We shall show that $\{\operatorname{cl} B_n : n \in \mathbb{N}\}$ is a convergent filter base. In fact, for each $n \in \mathbb{N}$ let $F_n = \operatorname{cl}\{x_m : m \geq n\}$. Then

$$F_n \subseteq \operatorname{cl} B_n \subseteq S_d(x_n, 2^{-n}) \cap S_q(x_n, 2^{-n})$$

for all $n \in \mathbb{N}$, so that there is a point $y \in \cap\{F_n : n \in \mathbb{N}\}$. Hence, the sequence $< x_n >$ has a subsequence $< x_{n(m)} >$ such that $q(y, x_{n(m)}) \to 0$ and $q(x_{n(m)}, y) \to 0$. Given a neighborhood U of y, there is $\epsilon > 0$ such that $\operatorname{cl} S_q(y, \epsilon) \subseteq U$. Assume that there is a sequence $< y_n >$ in X such that $y_n \in \operatorname{cl} B_n \setminus \operatorname{cl} S_q(y, \epsilon)$. Then there is a sequence $< z_n >$ in X such that $q(y_n, z_n) \to 0$, $q(x_n, z_n) \to 0$ and $q(y, z_n) \geq \epsilon$ for all $n \in \mathbb{N}$. Since q is a θ-metric, y is a cluster point of the sequence $< z_n >$, a contradiction. Therefore, the filter base $\{\operatorname{cl} B_n : n \in \mathbb{N}\}$ converges to y. We conclude that (X, T) is a complete Aronszajn space. ∎

THEOREM. *A space is completely metrizable if and only if it is a paracompact θ, weakly complete asymmetrizable space.*

Proof. Let (X, T) be a paracompact θ, weakly complete asymmetrizable space. Since a space is completely metrizable if and only if it is paracompact and complete Aronszajn [18], (X, T) is completely metrizable by Proposition 1. The converse is obvious. ∎

COROLLARY 1. *A space is completely metrizable if and only if it is a paracompact θ, Cauchy complete symmetrizable space.*

Proof. Every Cauchy complete symmetrizable space is a weakly complete symmetrizable space [11], [8]. ∎

COROLLARY 2. [14] *A space is completely metrizable space if and only if it is a paracompact weakly complete quasi-metrizable space.*

COROLLARY 3. *A space is completely metrizable if and only if it is a metrizable weakly complete semimetrizable space.*

EXAMPLE 1. The Pixley-Roy space over the reals provides an example of a Cauchy complete quasi-metrizable non complete Aronszajn space. Therefore it is not weakly complete asymmetrizable by Proposition 1.

EXAMPLE 2. The Michael line provides an example of a paracompact quasi-developable space [2] quasi-metrizable non-Moore space. Therefore, it is not weakly complete asymmetrizable.

Our next example shows that Cauchy complete symmetrizable cannot be replaced by Cauchy complete asymmetrizable in Corollary 1, not even for metrizable spaces.

EXAMPLE 3. Let $\mathbb{Q}$ be the set of rationals numbers and let d be the quasi-metric defined on $\mathbb{Q}$ by

$$\begin{aligned} d(x,y) &= 1 && \text{if } x > y \\ d(x,y) &= \min\{1, y-x\} && \text{if } x \le y. \end{aligned}$$

Then $T(d)$ is the restriction of the Sorgenfrey line to $\mathbb{Q}$. $(\mathbb{Q}, T(d))$ is a regular Hausdorff space with countable base, hence metrizable. Since it is a countable union of nowhere dense sets (singletons) it is not

Čech complete, so that it is not completely metrizable and, by Corollary 1, not Cauchy complete symmetrizable. However, d is a Cauchy complete quasi-metric for $(\mathbb{Q}, T(d))$.

EXAMPLE 4. Let W denote the set of all ordinals less than the first uncountable ordinal with the order topology. It is well known that W is a normal locally compact nonparacompact space. Furthermore W is a θ-space but not a γ-metrizable space [9, Example 4.12]. On the other hand, since every regular quasi-developable linearly ordered topological space is a paracompact space [3, Proposition 7 and Theorem 11], it follows that W is not quasi-developable. However, Proposition 2 (see below) shows that it admits a compatible strongly complete θ-metric, so that is a complete Aronszajn space by Proposition 1.

Recall that a space is said to be Čech complete [5] if it is a Tychonoff space which is a G_δ in some (equivalently, in any) of its (Hausdorff) compactifications.

PROPOSITION 2. *Each Čech complete θ-space admits a compatible strongly complete θ-metric.*

Proof. Let (X,T) be a Čech complete θ-space and let p be a θ-metric for (X,T). Since (X,T) is a Čech complete first countable space, it is easy to see that there is a strongly complete asymmetric q for (X,T) such that for each $x \in X$, $\{S_q(x,\epsilon) : \epsilon > 0\}$ is a neighborhood base at x. Hence $d = p \vee q$ is a strongly complete θ-metric for (X,T). ∎

COROLLARY 4. *Each Čech complete θ-space is a complete Aronszajn space.*

Proof. Apply Propositions 1 and 2. ∎

COROLLARY 5. ([15, Corollary 2.1]) *Each Čech complete quasi-developable space is a complete Aronszajn space.*

Proof. Each quasi-developable space is a θ-space [6]. ∎

Remark. Similarly to Proposition 2 one can show that each Čech complete γ-metrizable space admits a compatible strongly complete γ-metric.

REFERENCES

[1] ARHANGEL'SKII, A.V., Mappings and spaces, *Russian Math. Surveys,* **21** (1966), 115–162.

[2] BENNETT, H.R., On quasi-developable spaces, *General Topology and Appl.,* **1** (1971), 253–262.

[3] BENNETT, H.R. AND LUTZER, D.J., A note on weak θ-refinability, *General Topology and Appl.,* **2** (1972), 49–54.

[4] CEDER, J.G., Some generalizations of metric spaces, *Pacific J. Math.,* **11** (1961), 105–125.

[5] ENGELKING, R., "General Topology", Monografie Mat. 60 (PWN, Warsaw), 1977.

[6] FLETCHER, P., On wΔ-spaces, wσ-spaces and $\Sigma^{\#}$-spaces, *Pacific J. Math.,* **71** (1977), 419–428.

[7] FLETCHER, P. AND LINDGREN, W.F., "Quasi-Uniform Spaces", Marcel Dekker, New York, 1982.

[8] GALVIN, F. AND SHORE, S.D., Completeness in semimetric spaces, *Pacific J. Math.,* **113** (1984), 67–75.

[9] HODEL, R.E., Spaces defined by sequences of open covers which guarantee that certain sequences have cluster points, *Duke Math. J.,* **39** (1972), 253–263.

[10] KOFNER, J., On quasi-metrizability, *Toplogy Proc.,* **5** (1980), 111–138.

[11] MCAULEY, L.F., A relation between perfect separability, completeness and normality in semi-metric spaces, *Pacific J. Math.,* **6** (1956), 315–326.

[12] NEDEV, S.I., o-metrizable spaces, *Trans. Moscow Math. Soc.,* **24** (1971), 213–247.

[13] NIEMYTZKI, V.W., On the third axiom of metric space, *Trans. Amer. Math. Soc.,* **29** (1927), 507–513.

[14] ROMAGUERA, S., Left K-completeness in quasi-metric spaces, *Math. Nachr.,* **157** (1992), 15–23.

[15] ROMAGUERA, S., On complete Aronszajn quasi-metric spaces and subcompacteness, in "Papers on General Toplogy and Applications", Conference at Queens College 1992, Annals New York Acad. Sci. 728, 1994, 114–121.

[16] ROMAGUERA, S. AND SHORE, S.D., Developable spaces, developable distance functions, and g-developable spaces, preprint.

[17] WILSON, W.A., On quasi-metric spaces, *Amer. J. Math.,* **53** (1931), 675–684.

[18] WICKE, H.H. AND WORRELL, J.M., JR.,, Topological completeness of first countable Hausdorff spaces I, *Fund. Math.,* **75** (1972), 209–222.