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Natural Operations on Higher Order Tangent Bundles †

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We present a survey of some recent results about natural operations on the r-th order tangent bundle and similar objects.

The r-th order tangent bundle T^rM of a manifold M is the fundamental structure of higher order mechanics, [3], [4]. It has been clarified recently that several geometric properties of T^rM are related with the fact that the functor T^r preserves products. Since 1986 it has been known in differential geometry that all product preserving bundle functors are Weil functors and the algebraic properties of the corresponding Weil algebras have many important geometric consequences, see [11] for a survey. Moreover, the use of the Weilian approach is unavoidable, if one is interested in natural operations in the sense of [11]. And we have to underline that in many situations "natural" is nothing but a precisely defined synonym of a somewhat vague word "geometric".

In the present paper we collect the most interesting results from [11] and from some more recent papers on natural operations, which are closely related with the higher order mechanics. We intend to present a rounded survey of all basic relations between both subjects. In some cases our classification results imply that all natural operators were already constructed in the course of previous concrete research – but even this can be of some interest for applications. The first three sections of the present paper are devoted to the foundations of the Weilian approach. Then we discuss natural tensor fields of type (1,1), prolongation of the Frölicher-Nijehuis bracket, certain types of torsions, natural functions on T^*T^rM and natural operators transforming vector fields from a manifold to a Weil bundle. The last section is devoted to the idea of time-dependent Weil bundle and to natural (1,1)-tensor fields on such a bundle.

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All manifolds and maps are assumed to be infinitely differentiable.

1. Bundles of (k, r)-Velocities

In the higher order mechanics, the r-th order tangent bundle of a manifold M is defined to be the space $T^r M = J_0^r(\mathbb{R}, M)$ of all r-jets from $\mathbb{R}$ into M with source 0. More generally, one can construct the bundle of k-dimensional velocities of order r on M by setting $T_k^r M = J_0^r(\mathbb{R}^k, M)$. For every smooth map $f : M \to N$, one defines $T_k^r f : T_k^r M \to T_k^r N$ by

$$T_k^r f(j_0^r g) = j_0^r(f \circ g), \quad g : \mathbb{R}^k \to M.$$

Let $\mathcal{M}f$ be the category of all smooth manifolds and all smooth maps, $\mathcal{FM}$ be the category of all smooth fibered manifolds and their morphisms and $B : \mathcal{FM} \to \mathcal{M}f$ be the base functor. According to [11], a bundle functor on $\mathcal{M}f$ is a functor $F : \mathcal{M}f \to \mathcal{FM}$ satisfying $B \circ F = \mathrm{id}_{\mathcal{M}f}$ and the localization condition below. In other words, FM is a fibered manifold $p_M : FM \to M$ for every manifold M and Ff is an $\mathcal{FM}$-morphism $Ff : FM \to FN$ over f for every smooth map $f : M \to N$. The localization condition reads: if $i : U \to M$ is the inclusion of an open subset, then $FU = p_M^{-1}(U)$ and Fi is the inclusion $p_M^{-1}(U) \to FM$. Clearly, T_k^r is a bundle functor on $\mathcal{M}f$ and $T^r = T_1^r$. Moreover, T_k^r preserves products, i.e.

$$T_k^r(M \times N) = T_k^r M \times T_k^r N.$$

2. Product Preserving Bundle Functors

It has been clarified recently, see [11] for a survey, that the product preserving bundle functors on $\mathcal{M}f$ are something quite concrete – they coincide with the so-called bundles of infinitely near points introduced by A. Weil, [14]. Consider the addition and the multiplication of reals

$$a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Every product preserving bundle functor F on $\mathcal{M}f$ induces

$$Fa : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}, \qquad Fm : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}.$$

We have the following injection $i : \mathbb{R} \to F\mathbb{R}$. For every $x \in \mathbb{R}$, we consider the map $\widehat{x} : pt \to \mathbb{R}$, which transforms a one-point set pt into x. Since $F(pt) = pt$, we have $F\widehat{x} : pt \to F\mathbb{R}$ and we set $i(x) = F\widehat{x}(pt)$. One verifies easily that $F\mathbb{R}$ is a real vector space with the addition of vectors $Fa : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ and the multiplication of vectors by reals $Fm \circ (i \times \mathrm{id}_{F\mathbb{R}}) : \mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$. Moreover, the map $Fm : F\mathbb{R} \times F\mathbb{R} \to F\mathbb{R}$ is bilinear and endows $F\mathbb{R}$ with the structure of an algebra. The element $i(1) =: 1_{F\mathbb{R}}$ is the unit of $F\mathbb{R}$.

On the other hand, consider the algebra $\mathbb{R}[t_1, \dots, t_k]$ of all polynomials in k variables. Denote by $\langle t_1, \dots, t_k \rangle$ the ideal of all polynomials without absolute term and by $\langle t_1, \dots, t_k \rangle^r$ its r-th power, which is the ideal of all polynomials vanishing up to order $r - 1$ at the origin. By a Weil ideal we mean an ideal $\mathcal{A}$ satisfying $\langle t_1, \dots, t_k \rangle^{r+1} \subset \mathcal{A} \subset \langle t_1, \dots, t_k \rangle^2$. The factor algebra $A = \mathbb{R}[t_1, \dots, t_k]/\mathcal{A}$ is called a Weil algebra, the number k is said to be the width of A and the minimum of the r's is called the depth of A.

The following assertion is proved in [11], p. 308.

PROPOSITION 1. *$F\mathbb{R}$ is a Weil algebra for every product preserving bundle functor F on $\mathcal{M}f$.*

For example, in the case of the (k, r)-velocities functor T_k^r we have $T_k^r\mathbb{R} = \mathbb{R}[t_1, \dots, t_k]/\langle t_1, \dots, t_k \rangle^{r+1}$, which corresponds to the jet calculus. Furthermore, for the second iterated tangent functor TT we have $TT\mathbb{R} = \mathbb{R}[t_1, t_2]/\langle t_1^2, t_2^2 \rangle$. This reflects the well known fact that the elements of $TT\mathbb{R}^m$ are the equivalence classes of maps $g : \mathbb{R}^2 \to \mathbb{R}^m$ characterized by $g(0, 0)$, $\frac{\partial g(0,0)}{\partial t_1}$, $\frac{\partial g(0,0)}{\partial t_2}$, $\frac{\partial^2 g(0,0)}{\partial t_1 \partial t_2}$.

Conversely, every Weil algebra $A = \mathbb{R}[t_1, \dots, t_k]/\mathcal{A}$ determines a product preserving bundle functor on $\mathcal{M}f$ as follows. Consider the algebra $E(k)$ of germs of smooth functions on $\mathbb{R}^k$ at 0. We have a canonical injection $i : \mathbb{R}[t_1, \dots, t_k] \to E(k)$. The set $i(\mathcal{A})$ generates an ideal $\overline{\mathcal{A}} \subset E(k)$. Since $\mathcal{A}$ is a finitely generated ideal, we have $A = E(k)/\overline{\mathcal{A}}$ as well. Let M be a manifold. Two maps g, $h : \mathbb{R}^k \to M$, $g(0) = h(0) = x$, are said to be A-equivalent, if $\varphi \circ g - \varphi \circ h \in \widetilde{\mathcal{A}}$ for every germ φ of a smooth function on M at x. Such an equivalence class will be denoted by $j^A g$ and called an A-velocity on M. The point $g(0)$ is said to be the target of $j^A g$.

Denote by $T^A M$ the set of all A-velocities on M. One sees easily that $T^A\mathbb{R} = A$. The target map is a bundle projection $T^A M \to M$. Further, for every $f : M \to N$ we define $T^A f : T^A M \to T^A N$ by $T^A f(j^A g) = j^A(f \circ g)$. Then T^A is a product preserving bundle functor on $\mathcal{M}f$, which is called the Weil functor corresponding to A. The following assertion is proved in [11], p. 308.

PROPOSITION 2. *Every product preserving bundle functor F on $\mathcal{M}f$ is the Weil functor corresponding to the Weil algebra $F\mathbb{R}$, i.e. $F = T^{F\mathbb{R}}$.*

Remark 1. The Weilian approach to product preserving bundle functors on $\mathcal{M}f$ is unavoidable, if we are interested in their natural transformations. If B is another Weil algebra, then all natural transformations $T^A \to T^B$ are in bijection with the algebra homomorphisms $A \to B$, see [11], p. 307.

3. THE EXCHANGE MAP

Consider two velocities functors T^r_k and T^s_l. One verifies easily that the following procedure defines a natural equivalence κ of the iterated bundles $\kappa_M : T^s_l(T^r_k M) \to T^r_k(T^s_l M)$. For every $X = j^s_0\, g(t) \in T^s_l(T^r_k M)$, $t \in \mathbb{R}^l$, we have $g(t) = j^r_0(\tau \mapsto h(t,\tau))$, $\tau \in \mathbb{R}^k$, and we set $\kappa_M X = j^r_0(j^s_0(t \mapsto h(t,\tau))) \in T^r_k(T^s_l M)$. For $k = l = r = s = 1$ we obtain the canonical involution of TTM, [8].

Analogously we construct a canonical natural equivalence $\kappa_M : T^B(T^A M) \to T^A(T^B M)$ for every two Weil functors T^A and T^B. Let k or l be the width of A or B, respectively. Every $X \in T^B(T^A M)$ is of the form $X = j^B g(t)$, $t \in \mathbb{R}^l$, and $g(t) = j^A(\tau \mapsto h(t,\tau))$, $\tau \in \mathbb{R}^k$. Then we set

$$\kappa_M X = j^A(j^B(t \mapsto h(t,\tau))) \in T^A(T^B M)\,.$$

In particular, for $T^B = T$ we obtain a canonical natural equivalence

$$\kappa : TT^A \to T^A T\,. \tag{1}$$

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. A natural bundle over m-manifolds is a bundle functor $F : \mathcal{M}f_m \to \mathcal{FM}$, [13]. Obviously, the restriction of a bundle functor $\mathcal{M}f \to \mathcal{FM}$ to $\mathcal{M}f_m$ is a natural bundle over m-manifolds. For every vector field $X : M \to TM$, $m = \dim M$, one defines its flow prolongation $\mathcal{F}X : FM \to TFM$ by

$$\mathcal{F}X = \frac{\partial}{\partial t}|_0 F(\exp tX)\,.$$

In the case $F = T^A$, we can construct the induced map $T^A X : T^A M \to T^A TM$. Using (1), we obtain $\kappa_M^{-1} \circ T^A X : T^A M \to TT^A M$. The following result can be found in [11], p. 337.

PROPOSITION 3. *We have $\mathcal{T}^A X = \kappa_M^{-1} \circ T^A X$.*

4. Natural (1,1)-Tensor Fields

M. León and P. R. Rodrigues clarified the fundamental role of the canonical vertical operator $J_M : TT^rM \to TT^rM$, [4], which is a natural tensor field of type $(1,1)$ on M. For an arbitrary natural bundle F over m-manifolds, a natural tensor field of type $(1,1)$ on F is a system of $(1,1)$-tensor fields L_M on FM for every m-manifold M such that the diagram

$$\begin{array}{ccc} TFM & \xrightarrow{L_M} & TFM \\ {\scriptstyle TFf}\downarrow & & \downarrow{\scriptstyle TFf} \\ TFN & \xrightarrow{L_N} & TFN \end{array}$$

commutes for every local diffeomorphism $f : M \to N$.

All natural $(1,1)$-tensor fields on a Weil bundle $F = T^A$ can be constructed as follows. Consider the multiplication $\mu : \mathbb{R} \times TM \to TM$ of tangent vectors by reals. Taking into account $F\mathbb{R} = A$, we have $F\mu : A \times FTM \to FTM$. Using (1), we construct

$$\mathcal{F}\mu = \kappa_M^{-1} \circ F\mu \circ (\mathrm{id}_A \times \kappa_M) : A \times TFM \to TFM\,.$$

According to [12], each $L(a)_M := \mathcal{F}\mu(a, -) : TFM \to TFM$ is a $(1,1)$-tensor field on FM and the following assertion holds.

Proposition 4. *Every natural* $(1,1)$*-tensor field on* T^AM *is of the form* $L(a)_M$ *for all* $a \in A$.

By the definition of the product ab in A, we obtain immediately

Proposition 5. $L(a) \circ L(b) = L(ab)$ *for all* $a, b \in A$.

In particular, if 1_A is the unit of A, then $L(1_A)_M = \mathrm{id}_{TT^AM}$.

In the case of the r-th order tangent bundle T^rM, the canonical vertical operator J_M corresponds to the class $t + \langle t^{r+1} \rangle \in T_1^r\mathbb{R}$. Hence Proposition 5 yields $J_M^{r+1} = 0$. Moreover, Proposition 4 can be reformulated as follows.

Proposition 6. *All natural (1,1)-tensor fields on* T^rM *are linear combinations with constant coefficients of* $\mathrm{id}_{TT^rM}, J_M, \dots, J_M^r$.

5. Torsions

In [12], the natural $(1,1)$-tensor fields on $T^A M$ were used for introducing a kind of torsions. Every general connection $\Gamma : T^A M \to J^1 T^A M$ can be interpreted as a tangent valued 1-form γ on $T^A M$. For every $a \in A$, the Frölicher-Nijehuis bracket

$$[\gamma, L(a)_M]$$

is a tangent valued 2-form on $T^A M$, which is called the a-torsion of Γ. If Γ is the classical linear connection on $TM$and $L(a)_M = J_M$, we obtain the classical torsion of Γ.

In particular, for a general connection Γ on the r-th order tangent bundle $T^r M$ we have r torsions

$$[\gamma, J_M], [\gamma, J_M^2], \dots, [\gamma, J_M^r].$$

The case $r = 2$ was studied in detail in [12].

6. Prolongation of Tangent Valued Forms

A tangent valued k-form on M is an antisymmetric multilinear base preserving morphism

$$P : TM \times_M \cdots \times_M TM \to TM$$

with k terms on the left hand side. For $F = T^A$, we have

$$FP : FTM \times_{FM} \cdots \times_{FM} FTM \to FTM.$$

Analogously to Section 4 we construct $\mathcal{F}P = \kappa_M^{-1} \circ FP \circ (\kappa_M \times \cdots \times \kappa_M)$,

$$\mathcal{F}P : TFM \times_{FM} \cdots \times_{FM} TFM \to TFM.$$

This is a tangent valued k-form on FM, which is called the complete lift of P, [1]. Hence even $L(a)_M \circ \mathcal{F}P$ is such a form for all $a \in A$.

If Q is another tangent valued l-form on M, the Frölicher-Nijenhuis bracket $[P, Q]$ is a tangent valued $(k + l)$-form on M, [11], p.69. Using some ideas by J. Gancarzewicz, W. Mikulski and Z. Pogoda, [7], A. Cabras and the author, [1], deduced the following formula, in which ab is the product in A of $a, b \in A$.

Proposition 7. $[L(a)_M \circ \mathcal{F}P, L(b)_M \circ \mathcal{F}Q] = L(ab)_M \circ \mathcal{F}([P, Q])$.

In particular, for $a = b = 1_A$ we obtain $[\mathcal{F}P, \mathcal{F}Q] = \mathcal{F}([P,Q])$. For an arbitrary fibered manifold $Y \to M$, a general connection $\Gamma : Y \to J^1Y$ can be interpreted as a tangent valued 1-form γ on Y. Then $\mathcal{F}\gamma$ defines a connection $\mathcal{F}\Gamma$ on $FY \to FM$. The following assertion is a simple consequence of Proposition 7.

PROPOSITION 8. *Each a-torsion of $\mathcal{F}\Gamma$ vanishes, $a \in A$.*

Proof. We have $[\mathcal{F}\gamma, L(a)_Y] = [\mathcal{F}\gamma, L(a)_Y \circ \mathrm{id}_{TFY}] = L(a)_Y \circ \mathcal{F}([\gamma, \mathrm{id}_{TY}])$, but $[\gamma, \mathrm{id}_{TY}] = 0$. ∎

7. NATURAL FUNCTIONS

For every natural bundle G over m-manifolds, a natural function h is a system of functions $h_M : GM \to \mathbb{R}$ for every m-manifold M such that $h_M = h_N \circ Gf$ for every local diffeomorphism $f : M \to N$. For example, if we consider the Liouville form of T^*M as a function $TT^*M \to \mathbb{R}$, then this is a natural function on TT^*.

We are going to describe all natural functions on T^*T^r. Denote by $\lambda^1_M : T^rM \to TT^rM$ the generalized Liouville vector field, which is tangent for $k = 1$ to the reparametrizations

$$j^r_0 g(t) \mapsto j^r_0 g(kt), \quad g : \mathbb{R} \to M, \quad k \in \mathbb{R}.$$

Define $\widetilde{\lambda}^1_M : T^*T^rM \to \mathbb{R}$ by

$$\widetilde{\lambda}^1_M(w) = \langle \lambda^1_M(qw), w \rangle\,, \quad w \in T^*T^rM\,, \tag{2}$$

where $q : T^*T^rM \to T^rM$ is the bundle projection. Obviously, $\widetilde{\lambda}^1$ is a natural function on T^*T^r. Furthermore, we set

$$\lambda^k_M = J^{k-1}_M \circ \lambda^1_M\,, \quad k = 2, \dots, r$$

and we define $\widetilde{\lambda}^k_M : T^*T^rM \to \mathbb{R}$ in the same way as in (2). In [9], the following assertion is proved.

PROPOSITION 9. *All natural functions on T^*T^r are of the form $h(\widetilde{\lambda}^1, \dots, \widetilde{\lambda}^r)$ with arbitrary smooth function $h : \mathbb{R}^r \to \mathbb{R}$.*

In particular, for $r = 1$ we have a natural identification $T^*TM \approx TT^*M$, [11], p. 229. Thus, in this case all natural functions are of the form $h(\widetilde{\lambda})$ where h is any smooth function of one variable, λ_M is the Liouville vector field of TM and $\widetilde{\lambda}_M$ is constructed as in (2).

Remark 2. F. Cantrijn, M. Crampin, W. Sarlet and D. Saunders established a natural equivalence $T^*T^r \approx T^rT^*$ for every r, [2]. However, M. Doupovec and J. Kurek have deduced recently, [6] that there is no natural equivalence between $T^*T^1_k$ and $T^1_kT^*$ for $k \geq 2$. These results characterize an interesting difference between the r-th order tangent functor $T^r = T^r_1$ and the (k,r)-velocities functors $k \geq 2$.

8. Natural Operators on Vector Fields

The important role of the algebra $F\mathbb{R}$ in the geometry of a product preserving bundle functor F on $\mathcal{M}f$ can be clearly seen, when we study the natural operators transforming vector fields on a manifold M into vector fields on FM.

In general, let G be a natural bundle over m-manifolds and $C^\infty TM$ denote the set of all smooth vector fields on M. A natural operator $D : T \to TG$ is a system of maps $D_M : C^\infty TM \to C^\infty T(GM)$ for every m-manifold M such that

(i) $D_N(Tf \circ X \circ f^{-1}) = TGf \circ D_M X \circ (Gf)^{-1}$ for every $X \in C^\infty TM$ and every diffeomorphism $f : M \to N$,

(ii) $D_U(X|U) = (D_M X)|GU$ for every $X \in C^\infty TM$ and every open subset $U \subset M$,

(iii) every D_M is regular, i.e. D_M transforms every smoothly parametrized family of vector fields into a smoothly parametrized family.

For example, in the case G is the tangent bundle T, one constructs easily three such operators

1) the flow operator $\mathcal{T}$,

2) the vertical operator V, which extends every vector field X on M into a vector field $V_M(X)$ on TM by means of the translations in the individual fibers of TM,

3) the constant map $X \mapsto \lambda_M$ into the Liouville vector field. By [11], p. 356, all natural operators $T \to TT$ are of the form

$$c_1\mathcal{T} + c_2V + c_3\lambda\,, \qquad c_1, c_2, c_3 \in \mathbb{R}\,. \tag{3}$$

In the case of an arbitrary Weil functor T^A, we have to consider the group Aut A of all algebra automorphisms of A and its Lie algebra $\mathfrak{Aut}\,A$. Every $D \in \mathfrak{Aut}\,A$ is of the form $D = \frac{\partial}{\partial t}|_0\delta(t)$, $\delta(t) \in$ Aut A. By Remark 1, every $\delta(t)$ determines a natural transformation $\delta(t)_M : T^AM \to T^AM$. The induced vector field λ_M^D will be called the D-Liouville vector field on T^AM, $D \in \mathfrak{Aut}\,A$. On the other hand, if we take the flow prolongation $\mathcal{T}^AX$ of $X \in C^\infty TM$ and any $a \in A$, then $L(a)_M \circ \mathcal{T}^AX$ is also a vector field on T^AM. In [11], p. 356, the following result is deduced.

PROPOSITION 10. *All natural operators $T \to TT^A$ are of the form*

$$L(a) \circ \mathcal{T}^A + \lambda^D \quad \text{for all } a \in A \text{ and all } \quad D \in \mathfrak{Aut}\,A. \tag{4}$$

In (3), the classical Liouville vector field λ corresponds to the canonical basis of 1-dimensional space $\mathfrak{Aut}\,A$ and $V = J \circ \mathcal{T}$.

In the case of the r-th tangent functor T^r, (4) represents a $(2r+1)$-parameter family

$$c_0\mathcal{T}^r + c_1 J \circ \mathcal{T}^r + \dots + c_r J^r \circ \mathcal{T}^r + c_{r+1}\lambda^1 + \dots + c_{2r}\lambda^r, \quad c_0, \dots, c_{2r} \in \mathbb{R}.$$

9. Time-Dependent Weil Bundles

Generalizing the concept of time-dependent tangent bundle $TM \times \mathbb{R}$ from the non-autonomous dynamics, M. Doupovec and the author introduced the time-dependent Weil bundle $T^A_{\mathbb{R}}$ for every Weil algebra A. This is a bundle functor on $\mathcal{M}f$ defined by $T^A_{\mathbb{R}}M = T^AM \times \mathbb{R}$ for every manifold M and $T^A_{\mathbb{R}}f = T^Af \times \mathrm{id}_{\mathbb{R}}$ for every smooth map f.

All natural (1,1)-tensor fields on $T^A_{\mathbb{R}}$ are determined in [5]. Let dt be the canonical 1-form on $\mathbb{R}$. Every Liouville vector field λ_M^D defines a natural (1,1)-tensor field $\lambda_M^D \otimes dt$ on $T^A_{\mathbb{R}}M$, $D \in \mathfrak{Aut}\,A$. Moreover, every $L(a)_M$ induces a natural (1,1)-tensor field $\overline{L}(a)_M$ on $T^A_{\mathbb{R}}M$, $a \in A$, by means of the product structure on $T^AM \times \mathbb{R}$. In the same way, the identity of $T\mathbb{R}$ defines another natural tensor field $(\overline{\mathrm{id}}_{T\mathbb{R}})_M$ on $T^A_{\mathbb{R}}M$. In the following assertion, which is proved in [5], we do not distinguish between a real function $\mathbb{R} \to \mathbb{R}$ and its pullback $T^AM \times \mathbb{R} \to \mathbb{R}$.

PROPOSITION 11. *All natural (1,1)-tensor fields on $T^A_{\mathbb{R}}$ are linear combinations of*

$$\overline{L}(a),\ \lambda^D \otimes dt,\ \overline{\mathrm{id}}_{T\mathbb{R}}, \qquad a \in A, D \in \mathfrak{Aut}\,A,$$

the coefficients of which are arbitrary smooth functions on $\mathbb{R}$.

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