Continuity of Jordan Homomorphisms of Banach Algebras

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(Presented by J. Galé)

The long standing automatic continuity problems for homomorphisms of Banach algebras are:

QUESTION 1. If $T : A \longrightarrow B$ is a homomorphism from a Banach algebra A, such that $T(A)$ is dense in a semi-simple Banach algebra B, is T necessarily continuous?

QUESTION 2. Is every epimorphism from a Banach algebra onto a semi-prime Banach algebra necessarily continuous?

(See [1, Open questions (16) and (18), p. 463,464]).

Partial positive answers to both questions have been obtained by several authors. (See [1, p. 312-470] and [6]).

DEFINITION. A linear mapping $T : A \longrightarrow B$ between two algebras A and B is called a Jordan homomorphism if $T(xy + yx) = T(x)T(y) + T(y)T(x)$, for all x and y in A.

It is obvious that every ordinary homomorphism is a Jordan homomorphism. Also, it is well-known that the class of semi-simple algebras contains all semi-simple algebras.

We consider a more general form of the Question 1, as follows:

QUESTION 1'. Let A be a Banach algebra, and let B be a semi-prime Banach algebra. Is every Jordan homomorphism $T : A \longrightarrow B$ with dense range continuous?

Next we give some recently obtained results.

Let $\ell(\text{Socl } A)$ denote the left annihilator of the socle of a semi-prime algebra A. Firstly we have:

THEOREM 1. *Let $T : A \longrightarrow B$ be a homomorphism from a Banach algebra A into semi-simple Banach algebra B such that $T(A)$ is dense in B. If $\ell(\text{Socl } B) = \{0\}$, then T is continuous.*

More general results can be stated:

THEOREM 2. *Let A and B be Banach algebras, where B is semi-prime such that $\ell(\text{Socl } B) = \{0\}$. If $T : A \longrightarrow B$ is a homomorphism with dense range, then T is continuous.*

THEOREM 3. *Every Jordan homomorphism from a Banach algebra onto semi-simple Banach algebra is continuous.*

Detailed proofs, and other results may be found in the author's papers [2],[3],[4] and [5]. In general, the automatic continuity problems still await solution.

REFERENCES

[1] BACHAR, J.M., "Radical Banach Algebras and Automatic Continuity", Lecture Notes in Mathematics, 975, Springer-Verlag, 1983.

[2] DRĂGHIA, D.D., Norm characterizations of Jordan homomorphisms and Jordan derivations of Banach algebras, *Memoirs of the Faculty of Science,* Kochi University, Series A Mathematics, **14** (1993), 23 – 27.

[3] DRĂGHIA, D.D., Algebraic, spectral and topological characterizations of Jordan homomorphisms, *Memoirs of the Faculty of Science,* Kochi University, Series A Mathematics, **16** (1995), 49 – 56.

[4] DRĂGHIA, D.D., "Continuitate in Algebre Banach", Editura Didactică şi Pedagogică, Bucureşti, 1995.

[5] DRĂGHIA, D.D., Continuity of derivations and Jordan homomorphisms of semi-prime Banach algebras, *Studii si Cercetări Matematice,* **46** (3) (1994), 327 – 337.

[6] SINCLAIR, A.M., "Automatic Continuity of Linear Operators", Lecture Notes Series 21, Cambridge Press, 1976.