EXTRACTA MATHEMATICAE Vol. **9**, Núm. 3, 206 – 212 (1994)

A Ring to Describe Symbolic Expressions*

ANDRÉS BUJOSA AND REGINO CRIADO

Departamento de Matemática aplicada a las tecnologías de la información, E.T.S.I.T., Universidad Politécnica de Madrid

(Presented by Jesús M.F. Castillo)

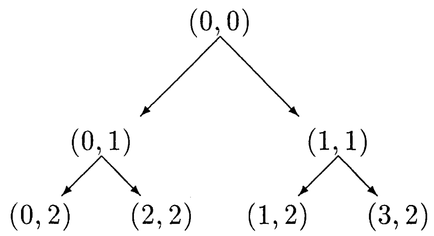
AMS *Subject Class.* (1991): 68Q40, 54E99 *Received January* 20, 1994

In Logic Programming it is usual the employ of tree-like structures as a way of representing programs and data (see [5]). We propose the use of a ring as a framework for data structures of the Logic Programming. The elements of our ring $\mathbb{J}_p$, introduced in [3], are called "p-tangles". This concept is a generalization of the concept of p-tree given in [1] and [2].

If p is a natural number greater than 1, we say that a pair of natural numbers (n,α) is a *p-path* if $0 \leq n < p^\alpha$. We denote by $\mathbb{P}_p$ the set of p-paths, i.e.

$$\mathbb{P}_p = \{(n,\alpha) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq n < p^\alpha\}.$$

Thus, if $p = 2$ we can represent graphically some 2-path as follows:



If (m,α) and (n,β) are elements of $\mathbb{P}_p$ we call *product of* (m,α) *and* (n,β) (in this order), to the pair $(m + n \cdot p^\alpha, \alpha + \beta)$. It is easy to see that if $(m,\alpha) \in \mathbb{P}_p$ and $(n,\beta) \in \mathbb{P}_p$, then $(m + n \cdot p^\alpha, \alpha + \beta) \in \mathbb{P}_p$ and that $\mathbb{P}_p$ together with this operation is a monoid. In the sequel, if x and y are elements of $\mathbb{P}_p$, we denote the product of x and y by xy, and we use the symbol **1** to denote the identity

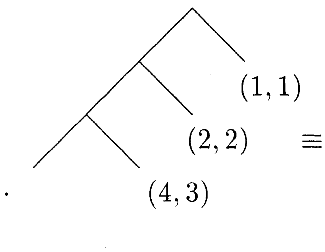
*This work was supported by Sistemas Expertos S.A. within ESPRIT II programme: European Declarative System under contract EP2025 (EDS).

element $(0,0)$. A property that we use in the sequel is the following: If (n, α) is an element of $\mathbb{P}_p$ and $\beta \in \mathbb{N}$ is such that $0 \leq \beta \leq \alpha$, then there exists a unique pair of elements of $\mathbb{P}_p(x, \gamma)$ and (y, β) such that $(x, \gamma)(y, \beta) = (n, \alpha)$.

If (n, α) is an element of $\mathbb{P}_p$, we call *module of* (n, α) to the real number $1/p^{\alpha}$, and we denote it by $|(n, \alpha)|$.

DEFINITION 1.1. We say that A is a *p-tangle* (or p-jungle) if A is a subset of $\mathbb{P}_p$. In the sequel we denote by $\mathbb{J}_p$ the set of all subsets of $\mathbb{P}_p$.

A binary tree B can be represented by the 2-tangle which elements are the p-paths associated to the leaves of B. By example,



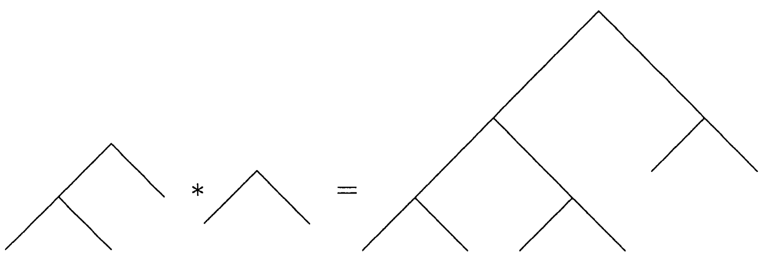
$$\equiv \left\{ (2^i, i+1) \mid i \in \mathbb{N} \right\}$$

Consider over $\mathbb{J}_p$ the "symmetrical difference operation" $A + B = (A - B) \cup (B - A)$ and the *product*

$$A * B = \{x \in \mathbb{P}_p \mid \text{card}(A \cdot B(x)) \text{ is odd } \},$$

where $A \cdot B(x)$ denotes the set $\{(a, b) \in A \times B \mid ab = x\}$.

$(\mathbb{J}_p, +, *)$ is a non commutative ring with identity. As an example of the meaning of the " * " operation we have that



$$\{LL, LR, R\} * \{L, R\} \; = \{LLL, LLR, LRL, LRR, RL, RR\}$$

$$\{(0,2),(2,2),(1,1)\} * \{(0,1),(1,1)\} = \{(0,3),(4,3),(2,3),(6,3),(1,2),(3,2)\}$$

DEFINITION 1.2. If $A \in \mathbb{J}_p$, we call *norm of* A, and we denote it by $\|A\|$, to:

$$\|A\| = \begin{cases} 0 & \text{if } A = 0, \\ \max\{|x| \in \mathbb{R} | x \in A\} & \text{if } A \neq \emptyset. \end{cases}$$

If $A, B, X, Y \in \mathbb{J}_p$ one has the following properties :

1. $\|A\| \in \mathbb{R} \ \wedge \ 0 \leq \|A\| \leq 1$ and $\|A\| = 0 \Leftrightarrow A = \emptyset$.
2. $\|A\| = 1 \Leftrightarrow 1 \in A$ and $\|A * B\| = \|A\| \, \|B\|$.
3. $(A \neq \emptyset \wedge A*X = A*Y) \Rightarrow X = Y$, and $(A \neq \emptyset \wedge X*A = Y*A) \Rightarrow X = Y$.
4. $\|A + B\| \leq \max(\{\|A\|, \|B\|\})$.
5. A is invertible if and only if $\|A\| = 1$. In that case $A^{-1} = \sum_{i=0}^{\infty}(A+\{1\})^i$
6. $d\colon \mathbb{J}_p \times \mathbb{J}_p \longrightarrow \mathbb{R}$, $(A, B) \rightsquigarrow \|A + B\|$ is an ultrametric over $\mathbb{J}_p$.

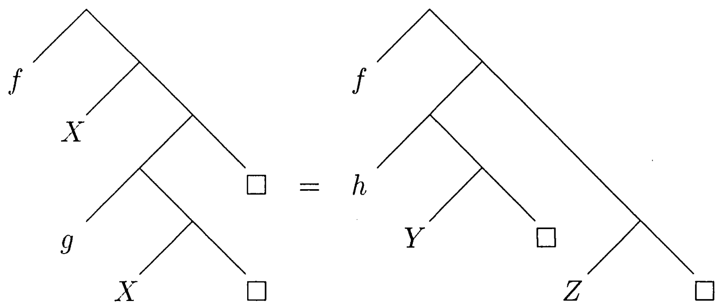
2. AN EXAMPLE

To illustrate the framework, let us suppose that it is required to unify the expressions "$f(X, g(X))$" and "$f(h(Y), Z)$", i.e., it is required to find a more general way to assign to the variables "X, Y, Z" some expressions such that if we replace this assignation in "$(X, g(X))''$ and "$f(h(Y), Z)$" we obtain the same expression.

In order to find an answer for this question it is need to determine the following set:

$$\{(X, Y, Z) \in Expressions^3 \mid f(X, g(X)) = f(h(Y), Z)\}.$$

Rewriting the equation $f(X, g(X)) = f(h(Y), Z)$ in a list form, it is obtained:



Now, it is possible to identify the atoms of the above list ("f","g", "h" and "$\square$ ") with some elements of the (left) $\mathbb{J}_2$– module $(\mathbb{J}_2)^4$:

$$\begin{array}{ccc} f & \longleftrightarrow & (1,0,0,0) \\ g & \longleftrightarrow & (0,1,0,0) \\ h & \longleftrightarrow & (0,0,1,0) \\ \square & \longleftrightarrow & (0,0,0,1) \end{array}$$

So, the set *Expressions* is included in the $\mathbb{J}_2$-module $(\mathbb{J}_2)^4$, and the above equation can be written

$$\{L\} \cdot f + \{RL, RRLRL\} \cdot X + \{RRLL\} \cdot g + \{RRR, RRLRR\} \cdot \square = \\ \{L\} \cdot f + \{RLL\} \cdot h + \{RLRL\} \cdot Y + \{RLRR, RRR\} \cdot \square + \{RRL\} \cdot Z.$$

Using the arithmetic representation of 2-paths, we have:

$$\{(0,1)\} \cdot f + \{(1,2),(11,5)\} \cdot X + \{(3,4)\} \cdot g + \{(7,3),(27,5)\} \cdot \square \\ \{(0,1)\} \cdot f + \{(1,3)\} \cdot h + \{(5,4)\} \cdot Y + \{(13,4),(7,3)\} \cdot \square + \{(3,3)\} \cdot Z.$$

Therefore, the equation $f(X, g(X)) = f(h(Y), Z)$ is equivalent to the linear equation:

$$\text{(I)} \quad \{(1,2),(11,5)\} \cdot X + \{(5,4)\} \cdot Y + \{(3,3)\} \cdot Z = \\ \{(1,3)\} \cdot h + \{(3,4)\} \cdot g + \{(13,4),(27,5)\} \cdot \square$$

Now, looking for a fundamental system of solution of the homogeneous equation

$$\{(1,2),(11,5)\} \cdot X + \{(5,4)\} \cdot Y + \{(3,3)\} \cdot Z = 0,$$

it is obtained that

$$\{(1,2),(11,5)\} \cdot X + \{(5,4)\} \cdot Y + \{(3,3)\} \cdot Z = 0$$

$\Leftrightarrow$

$$\exists W \in (\mathbb{J}_2)^4.. \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \{(1,2)\} \\ \{1\} \\ \{(5,4)\} \end{pmatrix} \cdot \begin{pmatrix} W \end{pmatrix}.$$

In order to find a particular solution of equation (I), one can suppose that the list X, Y, Z has the form:

$$X = X_h \cdot h + X_g \cdot g + X_\square \cdot \square\,; \\ Y = Y_h \cdot h + Y_g \cdot g + Y_\square \cdot \square\,; \\ Z = Z_h \cdot h = +Z_g \cdot g + Z_\square \cdot \square$$

where X_h, X_g, $X_\square$, Y_h, Y_g, $Y_\square$, Z_h, Z_g, y $Z_\square$ are elements of $\mathbb{J}_2$. As $\{h, g, \square\}$ is a free set of the module $(\mathbb{J}_2)^4$, the equation (I) is equivalent to the system:

(II)

$$\begin{cases} \{(1,2),(11,5)\} \cdot X_h + \{(5,4)\} \cdot Y_h + \{(3,3)\} \cdot Z_h &= \{(1,3)\} \\ \{(1,2),(11,5)\} \cdot X_g + \{(5,4)\} \cdot Y_g + \{(3,3)\} \cdot Z_g &= \{(3,4)\} \\ \{(1,2),(11,5)\} \cdot X_\square + \{(5,4)\} \cdot Y_\square + \{(3,3)\} \cdot Z_\square &= \{(13,4),(27,5)\} \end{cases}$$

A particular solution of (II) is:

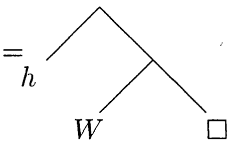
$$\begin{array}{lll} X_h = \{(0,1)\}, & Y_h = \emptyset, & Z_h = \{(1,3)\} \\ X_g = \emptyset, & Y_g = \emptyset, & Z_g = \{(0,1)\} \\ X_\square = \{(3,2)\}, & Y_\square = \emptyset, & Z_\square = \{(3,2),(13,4)\} \end{array}$$

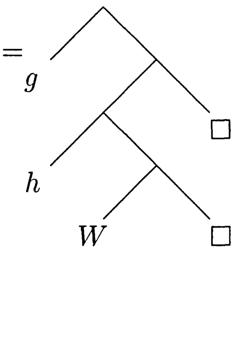
Thus, (X, Y, Z) is a solution of (I) if and only if there exists $W \in (\mathbb{J}_2)^4$ such that:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} =$$

$$\begin{pmatrix} \{(1,2)\} \\ \{1\} \\ \{(5,4)\} \end{pmatrix} \cdot \begin{pmatrix} W \end{pmatrix} + \begin{pmatrix} \{(0,1)\} \cdot h + \{(3,2)\} \cdot \square \\ 0 \\ \{(1,3)\} \cdot h + \{(0,1)\} \cdot g + \{(3,2),(13,4)\} \cdot \square \end{pmatrix}$$

Rewriting this solution in form of list:

$X =$ 

; $Y = W$; $Z =$ 

i.e. $X = h(W)$; $Y = W$; $Z = g(h(W))$.

3. Linear equations on $\mathbb{J}_p$

In this section some usual definitions of linear algebra are adapted to the context of p-tangles and a necessary and sufficient condition for the existence of solution of a linear equations system on $\mathbb{J}_p$ is obtained. The proofs of the results included in this section are in [4].

Note. $M_{n\times m}(\mathbb{J}_p)$ is the set of matrix of "n" files and "m" columns with coefficients in the ring $\mathbb{J}_p$. Also, if $A \in M_{n\times m}(\mathbb{J}_p)$, then A_i denotes the "i-file" of A.

Finally, if $A \in M_{n\times m}(\mathbb{J}_p)$ and $B \in M_{n\times 1}(\mathbb{J}_p)$ then $(A \mid B)$ is the amplified matrix of the system $A \cdot X = B$, so $(A \mid B) \in M_{n\times(m+1)}(\mathbb{J}_p)$.

Definition 3.1. Given $A \in M_{n\times m}(\mathbb{J}_p)$, A *is a non singular matrix* if $\forall X \in M_{m\times 1}(\mathbb{J}_p)(A \cdot X = 0 \Leftrightarrow X = 0)$. Also, $S \in M_{m\times k}(\mathbb{J}_p)$ *is a generator system of the solutions of the system* $A \cdot X = 0$ if

$$\forall X \in M_{m\times 1}(\mathbb{J}_p) \quad (A_i \cdot X = 0 \ \Leftrightarrow\ \exists Z \in M_{k\times 1}(\mathbb{J}_p)\ X = S \cdot Z)$$

Moreover, if S is a non singular matrix, S is a *fundamental system of solutions of* $A \cdot X = 0$.

Theorem 3.2. *If $A \in M_{n\times m}(\mathbb{J}_p)$ then any of the following properties is true:*

1. $\forall X \in M_{m\times 1}(\mathbb{J}_p), \quad A \cdot X = 0 \ \Leftrightarrow\ X = 0$
2. $\exists S \in M_{m\times k}(\mathbb{J}_p)$ *such that $k \leq m$, and S is a fundamental system of solutions of $A \cdot X = 0$. Also, if $0 \neq A$ then $k < m$.*

Lemma 3.3. *Let $a_1, a_2, \dots, a_n$ be elements of $\mathbb{J}_p$. The equation $\sum_i^n a_i * X_i = \{1\}$ has a solution if and only if there exists $i \in \{1, \dots, n\}$ such that a_i is invertible.*

Theorem 3.4. *If $A \in M_{n\times m}(\mathbb{J}_p)$ and $B \in M_{n\times 1}(\mathbb{J}_p)$ then the following conditions are equivalent:*

1. *The equation $A \cdot X = B$ has a solution*
2. *There exists a fundamental system of solutions of $(A \mid B) \cdot Y = 0$, and for any fundamental system of solutions of $(A \mid B) \cdot Y = 0$ the last file ("m-th file") has an invertible coefficient.*
3. *There exists a fundamental system of solutions of $(A \mid B) \cdot Y = 0$ such that its last file (m-th) file has some invertible coefficient.*

References

[1] Arriaga, F., Bujosa, A., Criado, R., A constructive definition of the space of infinite, p-trees: Topological properties, to appear in *Mathematica Japonica*.

[2] Arriaga, F., Bujosa, A., Criado, R., "LLULL: A language which comprises Logic, Functions and Processes", ESPRIT II $EP2025$: EUROPEAN DECLARATIVE SYSTEM (EDS). Ref: EDS.WP.9I.T001, (1989).

[3] Bujosa, A., Criado, R., p-Tangles: A ring with identity which contains the space of infinite p-trees, submitted.

[4] Bujosa, A., Criado, R., Syntactic elements of declarative programming: Symbolic linear equations, submitted.

[5] Lloyd, J.W., "Foundations of Logic Programming", (Second Extended Edition), Springer–Verlag, 1987.