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Steady and Asymptotic Analysis of the White–Metzner Fluid

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1. Introduction

The "spurt" phenomenon is a flow instability which occurs in presure–driven parallel shear flows of viscoelastic liquids. This phenomenon has been observed by Vinogradov et al. [3] in the flow of viscoelastic fluids through capillaires. They found that the volumetric flow rate increases dramatically at a critical stress that was independent of molecular Weight.

Recently, non–monotone (steady shear reponse) constitutive equations have been proposed to model this phenomenon.

The present paper analyzes systems of ordinary differential equations that approximate the dynamics of one–dimensional shear flow of highly elastic non–Newtonian fluids at low Reynolds number. The analysis reveals several distinctive phenomenas related to spurt: latency and normal stress oscillations. We study the dynamics of shear flow of White–Metzner model, presenting a description of non–Newtonian phenomena caused by a non-monotone relation between steady shear stress and shear strain rate.

2. Steady Flows

We examine parallel shear flow as depicted in [1]. By symmetry we need only to consider the flow on the interval $[-1/2, 0]$. The dimensionless time–dependent parallel shear flow equations are given by (see [1])

$$(2.1)\qquad \begin{cases} \hat{Re}\ v_t - \sigma_x = \theta v_{xx} + f & \text{(a)} \\ \sigma_t - [Z + \mu_{II}]v_x = -\frac{\sigma}{\lambda_{II}} & \text{(b)} \\ Z_t + \sigma v_x = -\frac{Z}{\lambda_{II}} & \text{(c)} \end{cases}$$

where $\hat{Re} = \frac{Re}{\omega}$, $\theta = \frac{1-\omega}{\omega}$, $(0 < \theta \leq 1)$ and

$$(2.2)\qquad \begin{aligned} \lambda_{II} &= \frac{We}{[1 + We^2 II]^{\beta}} \\ \eta_{II} &= \frac{1 + \lambda_{II}^2 II}{[1 + We^2 II]^{\alpha}} \end{aligned}$$

The boundary and initial conditions are given by

$$(2.3)\qquad \begin{array}{lll} v(-1/2, t) = 0, & v_x(0,t) = 0 & \text{(a)} \\ v(x,0) = v_0(x),\ \ \sigma(x,0) = \sigma_0(x) & Z(x,0) = Z_0(x) & \text{(b)} \end{array}$$

To conform with the symmetry, we require that $\sigma(0,0) = 0$. Then from (2.1)(a) we deduce that $\sigma(0,t) = 0$ for all time. We analyze steady state solution for a constant applied pressure gradient. The general steady state solutions denoted by $\bar{v}$, $\bar{\sigma}$ and $\bar{Z}$ satisfy

$$(2.4)\qquad \begin{aligned} \bar{\sigma} &= \frac{\eta_{\overline{II}} \bar{v}_x}{1 + \lambda_{\overline{II}}^2 \bar{v}_x^2} \\ \bar{Z} &= -\frac{\eta_{\overline{II}} \lambda_{\overline{II}}}{1 + \lambda_{\overline{II}}^2 \bar{v}_x^2} \end{aligned}$$

Therefore, we define the steady total shear stress by

$$(2.5)\qquad \bar{T} := \bar{\sigma} + \theta \bar{v}_x$$

and from (2.4) and (2.2) we deduce that $\bar{T}$ is given by

$$(2.6)\qquad \bar{T} := g(\bar{v}_x)$$

where

$$g(s) := \frac{s}{[1 + We^2 s^2]^\alpha} + \theta s \tag{2.7}$$

To study the behaviour of the flow, we need to know the properties of g. By symmetry, it suffices to consider $s \geq 0$.

When $0 \leq \alpha \leq 1/2$, the function g is strictly increasing, then from (2.6) we deduce that

$$\bar{v}_x = g^{-1}(\bar{T}) \tag{2.8}$$

The momentum equation, together with the boundary condition, implies that the steady total shear stress satisfies $\bar{T} = -fx$ for every $x \in [-1/2, 0]$ and the steady solution of system (2.1) is unique and it gives by $\bar{v}_x = g^{-1}(-fx)$. Then from (2.4) we deduce the stress components $\bar{\sigma}$ and $\bar{Z}$. Equivalently, a steady state solution $\bar{v}_x$ satisfies the cubic equation $P(\bar{v}_x) = 0$, where

$$P(s) := [1 + We^2 s^2]^\alpha (\theta s - \bar{T}) + s. \tag{2.9}$$

Now, let $1/2 < \alpha \leq 1$, there are two cases:

When $\theta > k(\alpha) := 2\left[1 + \frac{3}{2\alpha - 1}\right]^{-\alpha-1}$, the function g is strictly increasing and the system (2.1) admits a unique steady solution, but when $\theta < k(\alpha)$, the function g is not monotone.

g has a maximum at $s = s_M$ and a minimum at $s = s_m$ where

$$0 < s_M < s_{\inf} = \frac{1}{We}\sqrt{\frac{3}{2\alpha - 1}}; \quad s_{\inf} < s_m \quad \text{and} \quad g(s_m) < g(s_M).$$

Let $\bar{T}_M := g(s_M)$ and $\bar{T}_m := g(s_m)$, if $\bar{T}_m < -fx < \bar{T}_M$ then there are three distinct values of $\bar{v}_x$ that satisfy (2.6) for any particular x on the interval $[-1/2, 0]$. Consequently $\bar{v}_x$ can present jump discontinuities. Indeed, a steady solution must contain such a jump if the total stress $\bar{T}_{\text{wall}} = f/2$ at the wall exceeds the total stress $\bar{T}_M$ at the local maximum s_M.

Remark. In the case where $\alpha = 1$ and $\beta = 0$, we find the Johnson–Segalman models and the results studied in [2] with the Deborah number $\epsilon = \theta$.

3. Asymptotic Analysis

In this section we perform an asymptotic analysis of the spurt phenomenon for small θ and a negligible $\hat{Re}$. For Vinogradov et al.'s experiments [3] $\hat{Re} \approx 10^{-12}$, thus the inertia is totally negligible, and a system of phase space equations governing the dynamics at any particular point on the interval $[-1/2, 0]$ can be readily derived. When $\hat{Re} = 0$, the momentum equation in system (2.1) can be integrated, just as in the case of steady flows, to show that the total shear stress $T := \sigma + \theta \bar{v}_x$ coincides with the steady value $\bar{T}(x) = -fx$. Thus $T = \bar{T}(x)$ is a function of x only, even though σ and $\bar{v}_x$ are functions of both x and t. The remaining equations, for fixed values of x discribe an autonomous system of nonlinear ordinary differential equation, and are given by

$$\begin{cases} \sigma_t = [Z + \mu_{II}] \left(\frac{\bar{T}-\sigma}{\theta}\right) - \frac{\sigma}{\lambda_{II}} \\ Z_t = -\sigma \left(\frac{\bar{T}-\sigma}{\theta}\right) - \frac{Z}{\lambda_{II}} \end{cases} \tag{3.1}$$

where

$$II = \left(\frac{\bar{T} - \sigma}{\theta}\right)^2 .$$

The critical points of the above system are given by

$$\begin{cases} \left(Z + \mu_{II} + \frac{\theta}{\lambda_{II}}\right) \left(\frac{\sigma}{\bar{T}} - 1\right) + \frac{\theta}{\lambda_{II}} = 0 \\ \frac{\bar{T}^2}{\theta} \frac{\sigma}{\bar{T}} \left(\frac{\sigma}{\bar{T}} - 1\right) - \frac{Z}{\lambda_{II}} = 0 \end{cases} \tag{3.2}$$

These equations define two curves in the $\sigma - Z$ plane. The critical points are intersections of these curves. The equations of these curves are:

$$\begin{cases} Z = -\frac{\theta}{\lambda_{II}} \left[\frac{\sigma}{\sigma - \bar{T}} + \frac{\eta_{II}}{\theta}\right] \\ Z = \frac{We}{\theta} \frac{\sigma(\sigma - \bar{T})}{\left[1 + \left(\frac{We}{\theta}\right)^2 (\sigma - \bar{T})^2\right]^{\beta}} \end{cases}$$

The study of these functions shows that the critical points lie in the strip $0 < \sigma < \bar{T}$. Eliminating Z in the system (3.2) shows that the coordinates of the critical points satisfy the cubic equation $Q\left(\frac{\sigma}{\bar{T}}\right) = 0$, where

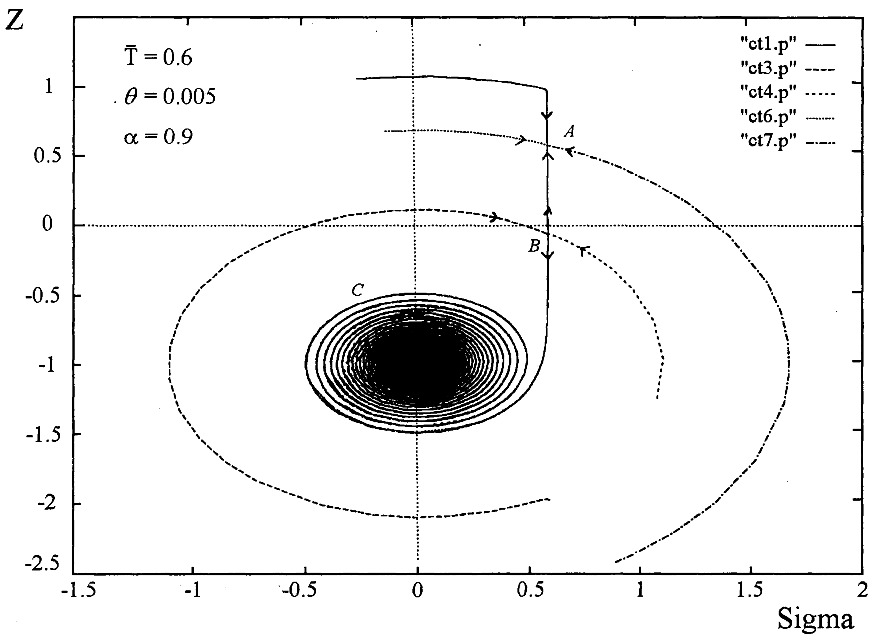

Figure 1. Phase portrait in the case of three critical points.

$$Q(\xi) := \xi \left[1 + \left(\frac{We\bar{T}}{\theta}\right)^2 (\xi - 1)^2\right]^\alpha + \frac{1}{\theta}(\xi - 1). \tag{3.3}$$

From (2.9) and (3.3) we show that

$$P(\bar{v}_x) = P\left(\frac{\bar{T} - \sigma}{\theta}\right) = -\bar{T}Q\left(\frac{\sigma}{\bar{T}}\right). \tag{3.4}$$

Thus each critical point of system (3.1) defines a steady state solution of system (2.1). Depending on the values of $\bar{T}$, α and θ, there are either one, two or three critical points. Consequently we have the following proposition.

PROPOSITION. *For all positions $x \in [-1/2, 0]$ there are many possibilities:*

(1) *If $0 \leq \alpha \leq 1/2$, there is a single critical point.*

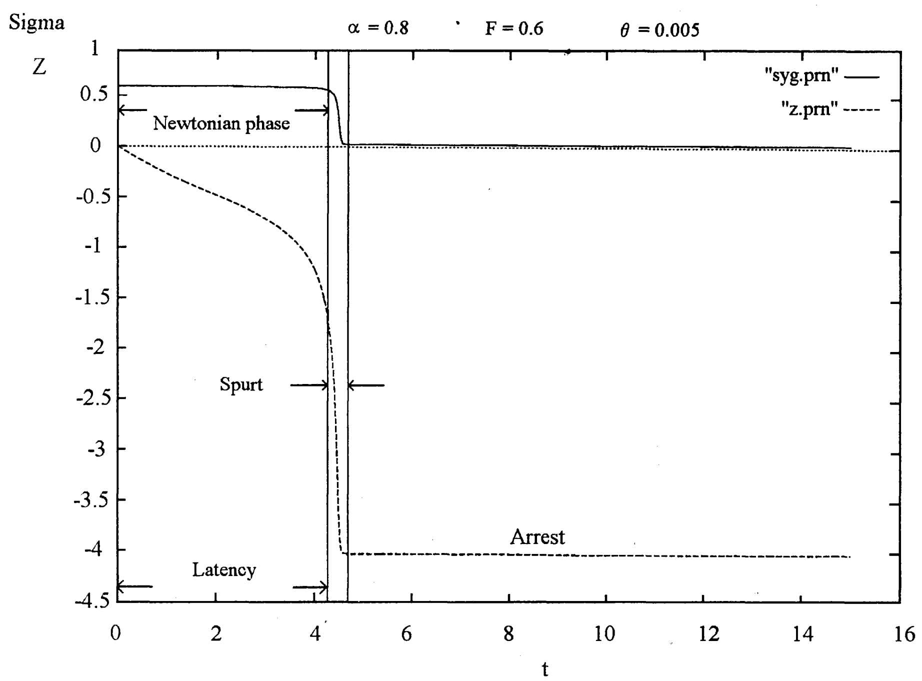

Figure 2. Results of stress sigma and Z vs. time as $\theta \to 0$.

(2) *If* $1/2 < \alpha \leq 1$, *there are two cases:*

(i) *There is a single critical point when* $\theta > k(\alpha)$.

(ii) *Let now* $\theta < k(\alpha)$, *in this case there are three possibilities:*

- *There is a single critical point when* $\bar{T}(x) < \bar{T}_m$.
- *There is a single critical point when* $\bar{T}_M < \bar{T}(x)$.
- *There are three critical points as shown in Fig. 1 when* $\bar{T}_m < \bar{T} < \bar{T}_M$.

To determine the qualitative structure of the dynamical system (3.1) we must study the nature of the critical points. The behaviour of orbits near a critical point depends on the linearization of (3.1) at this point, i.e., on the eigenvalues of the Jacobian of system (3.1) evaluated at the critical point. We omit the calculation because the expressions of the function of the relaxation time λ_{II} and the viscosity η_{II} are complicate. But when $\alpha = 1$ and $\beta = 0$, Malkus et al. [2] determine the dynamics of the approximating system completely. They show that the point A is an attracting node (called the classical attractor); B is a saddle point and C is either an attracting spiral point or an attracting node (called the spurt attractor). Fig. 1 reveals that in order to achieve a spurted solution the initial condition must lie in the buble region surrounding the point C.

As $\theta \rightarrow 0$, the dynamic evolution separates out in to three distinct regions as shown in Fig. 2 where numerical results for each of the stress are plotted as a function of time. In conclusion, we have investigated the dynamic behaviour of the spurt phenomenon for the White–Metzner model. We have found many similarities to the results obtained for the Johnson–Segalman model in [2]. The results provide an explanation of the spurt phenomenon observed experimentally. The key to understanding the dynamics of the approximating systems is fixing the location of the discontinuity the strain rate induced by the nonmonotone character of the steady shear stress versus strain rate.

References

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