EXTRACTA MATHEMATICAE Vol. 9, Núm. 3, 160 – 163 (1994)

The Polarisation Constant for JB^*-triples

P. MELLON

Department of Mathematics, University College, Belfield, Dublin 4, Ireland

(Presented by José M. Ansemil)

AMS *Subject Class.* (1991): 17C99 *Received December* 21, 1994

INTRODUCTION

In [2], Harris showed that for any J^*-algebra, Z, the polarisation constant $c(n, Z)$ satisfies

$$c(n, Z) \leq \frac{n^{n/2}(n+1)^{n+1/2}}{2^n n!} \qquad \text{for all } n \in \mathbb{N}.$$

A norm estimate for a derivative of the generalised Möbius transformation on JB^*-triples is presented in Corollary 3.6 of [4]. This estimate allows us to extend the result of Harris to the case of JB^*-triples, whose open unit balls have the same rich holomorphic structure as those of J^*-algebras. The JB^*-triples are, indeed, exactly those Banach spaces whose open unit balls have a transitive group of biholomorphic automorphisms.

NOTATION

Let E be a complex Banach space. Throughout, $\mathcal{L}(E)$ will denote the set of all continuous linear mappings on E and the spectrum of $T \in \mathcal{L}(E)$ is denoted $\sigma(T)$.

Let $\mathcal{L}_s^n(E)$ denote the space of all continuous symmetric n-linear mappings : $\underbrace{E \times \cdots \times E}_{n} \to \mathbb{C}$ and to each $p \in \mathcal{L}_s^n(E)$ denote by $\hat{p}$, the associated n-homogeneous polynomial $\hat{p}(x) = p(\underbrace{x, \ldots, x}_{n})$. Then

$$c(n, E) := \inf\{M : \|p\| \leq M\|\hat{p}\| \text{ for all } p \in \mathcal{L}_s^n(E)\}.$$

Note that, by the Hahn-Banach theorem, there is no loss of generality in considering complex-valued mappings in the above rather than Banach space-valued mappings.

The number $c(n, E)$ is called a polarisation constant for E and much effort has been taken to estimate $c(n, E)$ for various classes of spaces (*c.f.* [2], [5] and [6]). We refer to [1] for a general survey on this topic.

The class of JB^*-triples includes all C^*-algebras, J^*-algebras, JB^*-algebras and all complex Hilbert spaces. A JB^*-triple, Z, has a natural triple product $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$, which is symmetric and linear in the outer variables and antilinear in the inner variable and satisfies the following properties:

(i) the mapping $a \square a \in \mathcal{L}(Z)$ (where $a \square a(z) = \{a, a, z\}$, for all $z \in Z$) is Hermitian, for all $a \in Z$;

(ii) $\sigma(a \square a) \geq 0$ and $\|a \square a\| = \|a\|^2$ for all $a \in Z$;

(iii) $\{\alpha, \beta, \{x, y, z\}\} = \{\{\alpha, \beta, x\}, y, z\} - \{x, \{\beta, \alpha, y\}, z\} + \{x, y, \{\alpha, \beta, z\}\}$ for all α, β, x, y and z in Z.

For $a \in Z$, the mapping $Q(a) : Z \to Z$ given by $Q(a)(z) = \{a, z, a\}$ is antilinear.

An important class of elements in $\mathcal{L}(Z)$ are the Bergman operators

$$B(x, y) = I_Z - 2x \square y + Q(x)Q(y) \in \mathcal{L}(Z), \qquad \text{for all } x, y \in Z.$$

Let $B = \{z \in Z : \|z\| < 1\}$ and $\mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. The structure of $Aut(B)$ is examined in [3], to which we also refer the reader for any unexplained concepts in JB^*-triples.

Results

In [2] the author proves the above-mentioned estimates for the polarisation constants of a J^*-algebra. The key estimate in his proof is that

$$\| (g_z'(0))^{-1} \| \leq \frac{1}{1 - \|z\|^2}$$

where g_z is the Möbius transformation in $Aut(B)$ satisfying $g_z(0) = z$. This estimate is obvious in the case of J^*-algebras but is not at all transparent for JB^*-triples, even though it actually holds as an equality there.

For JB^*-triples

$$(g_z{}'(0))^{-1} = B(z, z)^{-1/2},$$

where $B(z,z)^{-1/2}$ is defined in the sense of the functional calculus.

It is interesting to note that while $B(z,z)^{-1/2}$ has positive spectrum it is not necessarily a Hermitian operator (*cf.* Example 4.5 in [4]). Nonetheless, it has been shown, Corollary 3.6 (i) in [4], that

$$\|B(z,z)^{-1/2}\| = \frac{1}{1-\|z\|^2}.$$

This will be the key to extending inequalities for J^*-algebras to the case of JB^*-triples.

Once this fact is noticed, Harris's proof extends directly to JB^*-triples. We consider it worthwhile to reproduce the proof again here, as its elegance is somewhat overshadowed by the many results presented in [2].

THEOREM. *Let Z be a JB^*-triple. Then*

$$c(n,Z) \leq \frac{n^{n/2}(n+1)^{n+1/2}}{2^n n!} \qquad \text{for all } n \in \mathbb{N}.$$

Proof. Let $h : B \to \overline{\mathbb{D}}$ be a holomorphic mapping. Fix z arbitrary in B and let $f = h \circ g_z$. By the Cauchy inequalities $\|f'(0)\| \leq 1$ and since $f'(0) = h'(z)B(z,z)^{1/2}$ it follows that

$$\|h'(z)\| \leq \|B(z,z)^{-1/2}\| = \frac{1}{1-\|z\|^2}.$$

Take $p \in \mathcal{L}_s^n(E)$ and let $\hat{p}$ be its associated n-homogeneous polynomial. We assume, without loss of generality, that $\|\hat{p}\| = 1$.

For any z and a in Z,

$$\hat{p}'(z)a = np(\underbrace{z,\dots,z}_{n-1},a).$$

In particular, for $z \in B$ and $a \in \overline{B}$ it follows from the above that

$$\|p(\underbrace{z,\dots,z}_{n-1},a)\| \leq \frac{1}{n(1-\|z\|^2)}.$$

Fix b in Z with $\|b\| = 1$, arbitrary. Replacing z by tb with $0 < t < 1$ we get

$$\|p(\underbrace{b,\dots,b}_{n-1},a)\| \leq \frac{1}{nt^{n-1}(1-t^2)}.$$

The right hand side is minimised by $t = \sqrt{\frac{n-1}{n+1}}$ and therefore

$$\|p(\underbrace{b, \dots, b}_{n-1}, a)\| \leq \frac{(n+1)^{\frac{n+1}{2}}}{2n(n-1)^{\frac{n-1}{2}}}.$$

Applying this $n-1$ times gives the result. ∎

In exactly the same way it can be shown that further estimates obtained by Harris in [2] for J^*-algebras also hold for JB^*-triples.

Let h be a holomorphic mapping defined on an open subset U of Z and let $x \in U$. Let $h^{(n)}(x)$ denote the symmetric continuous n-linear mapping on Z corresponding to the n-th order Fréchet derivative of h at x.

Using the same notation as in [2], let $M_{n,m}(r)$ be the supremum of $\|h^{(n)}(x)\|$ where the supremum is taken over all x in $\{z \in Z : \|z\| = r\}$ and over all holomorphic mappings $h : B \to \overline{\mathbb{D}}$ satisfying $h(0) = 0, h'(0) = 0, \dots, h^{(m-1)}(0) = 0$ (where, for $m = 0$, no condition is imposed).

Mimicking [2], we get the following:

COROLLARY. *For Z any JB^*-triple*

$$M_{n,m}(r) \leq \left(\frac{m^m(m+1)^{m+1}}{2^{2n}(m-n+1)^{m-n+1}(m-n)^{m-n}}\right)^{1/2} r^{m-n}$$

for all r with $0 \leq r \leq \sqrt{1-n/m}\sqrt{1-n/(m+1)}$ and $1 \leq n < m$.

REFERENCES

[1] DINEEN, S. , "Complex Analysis in Infinite Dimensional Spaces", In Preparation.

[2] HARRIS, L.A. , Bounds on the derivatives of holomorphic functions of vectors, in "Colloque d'analyse, Rio de Janeiro, 1972", ed. L. Nachbin, Act. Sc. et Ind. 1367, 145–163, Hermann, Paris, 1975.

[3] KAUP, W. , A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Math. Z.* **183** (1983), 503–529.

[4] KAUP, W. , Hermitian Jordan triple systems and the automorphisms of bounded symmetric domains, in "Proceedings of Third International Conference on non Associative Algebra and its Applications", Oviedo, Spain, July 12-17, 1993.

[5] TONGE, A. , Polarization and the complex Grothendieck inequality. *Math. Proc. Camb. Phil. Soc.* **95** (1984), 313–318.

[6] SARANTOPOULOS, I. , Estimates for polynomial norms on $L^p(\mu)$ spaces. *Math. Proc. Camb. Phil. Soc.* **99** (1986), 263–271.