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Topological Localization in Fréchet Algebras *

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INTRODUCTION

Let F be a closed subset of a normal topological space X. The construction of the algebra $\mathscr{C}(F)$, of all complex-valued continuous functions on F, when one knows the algebra $\mathscr{C}(X)$ follows from Tietze's theorem: since the restriction morphism $\mathscr{C}(X) \to \mathscr{C}(F)$ is surjective, one has

$$\mathscr{C}(F) = \mathscr{C}(X)/J \ , \tag{1}$$

where J denote the ideal of all continuous functions on X vanishing on F. Analogously, if C is a closed submanifold of a Hausdorff σ-compact manifold V of class r, then the restriction morphism $\mathscr{C}^r(V) \to \mathscr{C}^r(C)$ is surjective, thus

$$\mathscr{C}^r(C) = \mathscr{C}^r(V)/I \ , \tag{2}$$

where I denote the ideal of all differentiable functions on V vanishing on C.

If U is an open set in a topological space X, the construction of $\mathscr{C}(U)$ follows from a result of Hager ([8]) (at least when U is a cozero-set): each continuous function on U is a quotient of two functions defined on X, where the denominator does not vanish at any point of U; that is to say

$$\mathscr{C}(U) = \mathscr{C}(X)_S = \{f/g \ : \ f \in \mathscr{C}(X), g \in S\} \ , \tag{3}$$

where $\mathscr{C}(X)_S$ denotes the localization (or ring of fractions) of $\mathscr{C}(X)$ with respect to the multiplicatively closed subset $S = \{f \in \mathscr{C}(X) \ : \ 0 \notin f(U)\}$. Analogously, if U is an open set in a Hausdorff σ-compact differentiable manifold V of class r, then Muñoz and Ortega proved in [13] that any differentiable function on U is a quotient of two differentiable functions on V; i.e., taking $S = \{f \in \mathscr{C}^r(V) \ : \ 0 \notin f(U)\}$,

$$\mathscr{C}^r(U) = \mathscr{C}^r(V)_S \ . \tag{4}$$

However, the algebras involved in these results are topological algebras and the isomorphisms (1) and (2) are homeomorphisms (the quotient algebra of a topological algebra by a closed ideal has a natural quotient topology). In the paper we define

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a natural topology on the localization of a topological algebra with respect to a multiplicatively closed set and we prove that (3) and (4) are homeomorphisms.

In general, let A be a symmetric Fréchet algebra, let X be the topological spectrum of A and let $\tilde{A}$ be the structural sheaf of algebras on X. For U an open set in X, we denote by A_U the localization of A with respect to the multiplicatively closed subset $S_U = \{a \in A : 0 \notin a(U)\}$. The main result of the paper may be stated as follows:

THEOREM *If A is strictly regular and U is a cozero-set, then the natural algebra morphism $A_U \longrightarrow \tilde{A}(U)$ is an isomorphism. Moreover, A_U is a strictly regular symmetric Fréchet algebra with spectrum U.*

The isomorphism $A_U = \tilde{A}(U)$ was proved by Muñoz and Ortega ([13]) in the semisimple case, and by Brooks ([3]) when A is semisimple and $U = X$.

1. Real Fréchet Algebras

DEFINITIONS 1.1. By a *topological $\mathbb{K}$-algebra* ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) we shall mean a $\mathbb{K}$-algebra A endowed with a topology such that A is a topological $\mathbb{K}$-vector space, the ring multiplication $A \times A \longrightarrow A$ is continuous, and A has continuous inverse. Let A, B be topological $\mathbb{K}$-algebras. The set of all continuous $\mathbb{K}$-algebra morphisms from A to B is denoted by $\mathrm{Hom}_{\mathbb{K}}(A, B)$. A topological $\mathbb{K}$-algebra A is said to be *Fréchet* when it is Hausdorff, complete and its topology is defined by a countable increasing sequence of $\mathbb{K}$-multiplicative semi-norms.

NOTE 1.2. Let A be a topological $\mathbb{K}$-algebra, let B be a $\mathbb{K}$-algebra and let $f : A \longrightarrow B$ be a $\mathbb{K}$-algebra morphism. There exists on B the finest topology such that B is a topological $\mathbb{K}$-algebra and the morphism f is continuous; this topology is said to be the *final algebra topology* defined on B by f.

If X is a Hausdorff, locally compact and σ-compact space, then $\mathscr{C}(X)$ is a Fréchet $\mathbb{R}$-algebra. If V is a $\mathscr{C}^r$-manifold, then the algebra $\mathscr{C}^r(V)$ of all real-valued functions on V of class r, endowed with the topology of uniform convergence on compact sets of the functions and all of their derivatives up to order r, is a Fréchet $\mathbb{R}$-algebra (IV.4.2 of [11]).

If I is a closed ideal of a Fréchet $\mathbb{K}$-algebra A, then the quotient algebra A/I, endowed with the quotient topology, is a Fréchet $\mathbb{K}$-algebra.

DEFINITIONS 1.3. Let A be a topological $\mathbb{K}$-algebra and let $X = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. Each element $a \in A$ defines a $\mathbb{K}$-valued map on X: $a : X \longrightarrow \mathbb{K}$, $x \mapsto a(x) = x(a)$. The *topological spectrum* (or just spectrum, if no confusion is possible) of A is the set X endowed with the initial topology defined by these maps (the *Gelfand topology*). It is denoted by $\mathrm{Spec}_t\, A$ and it is a completely regular Hausdorff space. If $a \in A$, then we set $(a)_0 = \{x \in X : a(x) = 0\}$, and the zeroes of an ideal I of A is defined to be the closed set $(I)_0 = \{x \in X : a(x) = 0 \text{ for all } a \in I\}$. An open set U in X is said to be a *cozero-set* if there exists $a \in A$ such that $U = \{x \in X : a(x) \neq 0\}$. A is said to be *regular* when cozero-sets form a basis for the topology of X.

By definition, each element $a \in A$ defines a continuous $\mathbb{K}$-valued function on X. So, we obtain the *spectral representation* morphism $A \longrightarrow \mathscr{C}(X, \mathbb{K})$, A is said to be *semisimple* when this morphism is injective (i.e., when each element of A may be identified with a continuous function on the spectrum).

EXAMPLES 1.4. (a) If I is an ideal of a topological $\mathbb{K}$-algebra A, then A/I, endowed with the final algebra topology defined by the canonical morphism $A \longrightarrow A/I$ (see 1.2), is a topological $\mathbb{K}$-algebra such that $\operatorname{Spec}_t(A/I) = (I)_0$. Therefore, A/I is regular when so is A.

(b) Let V be a $\mathscr{C}^r$-manifold. Each point $x \in V$ defines a continuous morphism $\mathscr{C}^r(V) \longrightarrow \mathbb{R}$, $f \mapsto f(x)$, and so we get a continuous map $i : V \longrightarrow \operatorname{Spec}_t \mathscr{C}^r(V)$. It is well-known that $i : V \longrightarrow i(V)$ is a homeomorphism (differentiable functions separate points and closed sets), and it may be proved that i is surjective. It follows that $V = \operatorname{Spec}_t \mathscr{C}^r(V)$ and $\mathscr{C}^r(V)$ is a semisimple regular Fréchet $\mathbb{R}$-algebra.

(c) There are important non-semisimple regular Fréchet algebras; the simplest one: $\mathscr{C}^\infty(\mathbb{R})/(x^2)$.

(d) If X is a Hausdorff, locally compact and σ-compact space, then $X = \operatorname{Spec}_t \mathscr{C}(X)$, so that $\mathscr{C}(X)$ is semisimple and regular.

DEFINITION 1.5. A Fréchet $\mathbb{R}$-algebra A is said to be *rational* if any continuous $\mathbb{R}$-algebra morphism $A \longrightarrow \mathbb{C}$ is real-valued: $\operatorname{Hom}_{\mathbb{R}}(A, \mathbb{R}) = \operatorname{Hom}_{\mathbb{R}}(A, \mathbb{C})$ (i.e., rational Fréchet algebras are just real Fréchet algebras with no imaginary points).

EXAMPLES 1.6. (a) Finite direct products, localizations (see Definition 2.2), quotients by closed ideals and completed tensorial products (with the π-topology, see X.3.1 of [10]) of rational Fréchet algebras are rational Fréchet algebras.

(b) Let A be a real Fréchet algebra. Is it easy to prove that A is rational if and only if $1 + a^2$ is invertible in A for any $a \in A$. Hence, $\mathscr{C}^r(V)$ and $\mathscr{C}(X)$ (when the space X is Hausdorff, locally compact and σ-compact) are rational Fréchet algebras.

(c) Let $\{A_i\,,\, \{f_{ij}, i \le j\}\,,\, i \in \mathbb{N}\}$ be a countable projective system of rational Fréchet algebras and let $A = \varprojlim A_i$. It follows from (b) that A is rational.

DEFINITION 1.7. By a symmetric Fréchet algebra we shall mean a complex Fréchet algebra B endowed with a continuous involution $B \xrightarrow{*} B$ such that any continuous $\mathbb{C}$-algebra morphism $B \longrightarrow \mathbb{C}$ commutes with the respective involutions. Morphisms of symmetric Fréchet algebras are defined to be continuous $\mathbb{C}$-algebra morphisms commuting with the involutions.

NOTE 1.8. If B is a symmetric Fréchet algebra, then $B_h = \{b \in B : b^* = b\}$ is a rational Fréchet algebra such that $B = B_h \oplus i \cdot B_h$. Conversely, if A is a Fréchet $\mathbb{R}$-algebra, then $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C} = A \oplus i \cdot A$, endowed with the direct sum topology, is a Fréchet $\mathbb{C}$-algebra with a continuous involution, $(a + i \cdot b)^* = a - i \cdot b$, and $\operatorname{Hom}_{\mathbb{R}}(A, \mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, \mathbb{C})$. Hence, if A is rational, then $A_{\mathbb{C}}$ is symmetric and $\operatorname{Spec}_t A = \operatorname{Spec}_t A_{\mathbb{C}}$, so that A is regular (resp. semisimple) if and only if so is $A_{\mathbb{C}}$.

Symmetric Fréchet algebras have been studied by many authors and there are well-known results about them (see [12] and [13]). Such results, according to 1.8, remain valid for rational Fréchet algebras.

2. Topological Localization

Let S be a *multiplicatively closed* subset of a topological $\mathbb{K}$-algebra A. A_S will denote the *localization* (or *ring of fractions*) of A with respect to S, and $h : A \longrightarrow A_S$, $h(a) = a/1$, will be the canonical morphism of A into A_S (see [1]).

Examples 2.1. (a) Let U be a cozero-set in a topological space X. Then we have $\mathscr{C}(U) = \mathscr{C}(X)_S$, where $S = \{f \in \mathscr{C}(X) : 0 \notin f(U)\}$ (see [8]).

(b) Let U be an open set in a $\mathscr{C}^r$-manifold V. Then we have $\mathscr{C}^r(U) = \mathscr{C}^r(V)_S$, where $S = \{f \in \mathscr{C}^r(V) : 0 \notin f(U)\}$ (see [13]).

(c) Let U be an open set in the spectrum of a semisimple, regular and rational Fréchet algebra A. If $\mathscr{A}(U)$ denotes the algebra of all real-valued continuous functions on U that locally coincide with functions defined by elements of A, then we have $\mathscr{A}(U) = A_S$, where $S = \{a \in A : 0 \notin a(U)\}$ (see [13]).

Definition 2.2. We define the *topological localization* of A with respect to S to be the algebra A_S endowed with the final algebra topology defined by h (see 1.2).

It is clear that A_S is a topological $\mathbb{K}$-algebra characterized by the following universal property: "*If $f : A \longrightarrow B$ is a continuous $\mathbb{K}$-algebra morphism such that $f(s)$ is invertible in B for all $s \in S$, then there exists a unique continuous $\mathbb{K}$-algebra morphism $\bar{f} : A_S \longrightarrow B$ such that $f = \bar{f} \circ h$*". It is easy to see that $\bar{f}(a/s) = f(a) \cdot [f(s)]^{-1}$. As consequence of this universal property, we have:

Lemma 2.3. *If U is a cozero-set in the spectrum of the topological algebra A and $S = \{a \in A : a(x) \neq 0$ for all $x \in U\}$, then $\operatorname{Spec}_t A_S = U$.*

On the one hand, there exists a "saturated" multiplicatively closed subset Z of A (i.e.: $a \cdot b \in Z \Leftrightarrow a, b \in Z$) containing S such that $A_S \longrightarrow A_Z$ is an isomorphism (and, by the universal property, an homeomorphism): $Z = \{a \in A : a/1$ is invertible in $A_S\}$. Hence, we may assume that any multiplicatively closed subset is saturated.

On the other hand, the canonical map $\pi : A \times S \longrightarrow A_S$, $\pi(a, s) = a/s$ is continuous, so that, if τ_c is the final topology defined in A_S by π, then $\tau_c \geq \tau_s$ (where τ_s denote the topology of the topological $\mathbb{K}$-algebra A_S). Furthermore, the morphism $h : A \longrightarrow (A_S, \tau_c)$ is also continuous. Hence, if (A_S, τ_c) is a topological $\mathbb{K}$-algebra, then $\tau_s \geq \tau_c$, and we may conclude that $\tau_c = \tau_s$. We have:

Theorem 2.4. *Let S be a saturated multiplicatively closed subset of a topological algebra A. If the canonical map $\pi : A \times S \longrightarrow A_S$ has a continuous section with respect to the topology τ_c, then (A_S, τ_c) is a topological algebra.*

As consequence, if τ is a topology on A_S such that the map $\pi : A \times S \longrightarrow (A_S, \tau)$ is continuous and has a continuous section, then $\tau = \tau_c = \tau_s$.

3. Strictly Regular Algebras

Let A be a topological $\mathbb{K}$-algebra and let $X = \mathrm{Spec}_t A$. If U is an open set in X, we denote by A_U the localization of A with respect to $S_U = \{a \in A : 0 \notin a(U)\}$. The functor $U \mapsto A_U$ is a presheaf of algebras on X whose associated sheaf will be denoted by $\tilde{A}$ (= the structural sheaf on X), so that we have a natural morphism $A_U \longrightarrow \tilde{A}(U)$. If $x \in X$ and A_x denotes the localization of A with respect to $S_x = \{a \in A : a(x) \neq 0\}$, then the stalk of $\tilde{A}$ at x is just A_x. Therefore, if $a \in A$, the germ a_x of a at x is the image of a by the morphism $A \longrightarrow A_x$. The support of a is defined to be the closed subset $|a| = \{x \in X : a_x \neq 0\}$ (see [6], [7], [10]).

Examples 3.1. (a) Let X be a completely regular topological space and let $A = \mathscr{C}(X)$. If U is an open subset of X and $A_U = A_S$, $S = \{f \in A : 0 \notin f(U)\}$, then we have a natural morphism $A_U \longrightarrow \mathscr{C}(U)$ (an isomorphism when U is cozero) and we obtain a sheaf morphism $\tilde{A} \longrightarrow \mathscr{C}$. Since cozero-sets form a basis for the topology of X, we have $\tilde{A} = \mathscr{C}$, so that example 2.1.(a) shows that $A_U = \tilde{A}(U)$ whenever U is a cozero-set.

(b) Let V be a $\mathscr{C}^r$-manifold and let $A = \mathscr{C}^r(V)$. Then $\tilde{A} = \mathscr{C}^r$, so that example 2.1.(b) shows that $A_U = \tilde{A}(U)$ for any open set U in V.

(c) Let A be a semisimple, regular and rational Fréchet algebra. Then we have a sheaf of algebras $\mathscr{A}$ on $\mathrm{Spec}_t A$ (see 2.1.(c)). If U is an open set in $\mathrm{Spec}_t A$ and $s \in S_U$, then $1/s$ is a function on U that locally coincides with functions defined by elements of A ([13], Theorem 18), so that the natural morphism $A \longrightarrow \mathscr{A}(U)$ factors through A_U and we obtain a sheaf morphism $\tilde{A} \longrightarrow \mathscr{A}$; in fact, it is an isomorphism. Hence, example 2.1.(c) shows that $A_U = \tilde{A}(U)$ for any cozero-set U in $\mathrm{Spec}_t A$.

(d) For a certain class of rational Fréchet algebras (containing any quotient of $\mathscr{C}^r(V)$ by a closed ideal) Ortega proved in [16] that $U \mapsto A_U$ is a sheaf on $\mathrm{Spec}_t A$. Therefore, for such algebras, we have $A_U = \tilde{A}(U)$ for any open set U in $\mathrm{Spec}_t A$ (in this case, any open subset of $\mathrm{Spec}_t A$ is a cozero-set).

Definition 3.2. Let A be a topological $\mathbb{K}$-algebra and let $X = \mathrm{Spec}_t A$. If $Y \subseteq X$, then $N_Y = \{a \in A : a_x = 0 \text{ for all } x \in Y\}$ is clearly an ideal of A and its closure $W_Y = \overline{N_Y}$ is said to be the *Whitney ideal* of Y. One says that A is *strictly regular* if we have $(N_F)_0 = F$ for any closed subset F of X, i.e., if A (as a family of sections of the sheaf $\tilde{A}$) separates points and closed sets.

Any strictly regular algebra is regular. Moreover, if A is semisimple and regular, then it is readily shown that the germ a_x of a at a point $x \in X$ is zero if and only if the map $a : X \longrightarrow \mathbb{K}$ vanishes on a neighbourhood of x. Hence, for semisimple algebras, strict regularity coincides with regularity. $\mathscr{C}^r(V)$ and $\mathscr{C}(X)$ (when the space X is Hausdorff, locally compact and σ-compact) are strictly regular Fréchet algebras.

From the properties of the topological spectrum (1.4.(a) and 2.3), it follows that localizations and quotients of strictly regular algebras also are strictly regular. Therefore, any quotient of $\mathscr{C}^r(V)$ by a closed ideal is a strictly regular rational Fréchet algebra.

We point out the following properties about strictly regular algebras:

THEOREM 3.3. *Let A be a strictly regular rational Fréchet algebra and let $X = \operatorname{Spec}_t A$. Then:*
(a) *$W_X = N_X = 0$, i.e., the morphism $A \longrightarrow \tilde{A}(X)$ is injective.*
(b) *If F and C are closed sets in X such that $F \subseteq$ (interior of C), then $W_C \subseteq N_F$.*
(c) *If F_1 and F_2 are disjoint closed sets in X, then $N_{F_1} + N_{F_2} = A$.*
(d) *If F is a closed set in X, then W_F is contained in any other closed ideal I of A such that $F = (I)_0$.*

THEOREM 3.4. (Partitions of unity) *Let A be a strictly regular rational Fréchet algebra and let $X = \operatorname{Spec}_t A$. If $\{U_i\}_{i \in \mathbb{N}}$ is a countable open covering of X and $\{C_{i,n}\}_{i,n \in \mathbb{N}}$ is a double sequence of non-negative real numbers, then there exists a sequence $(a_i)_{i \in \mathbb{N}}$ in A such that the family $\{|a| \, : \, i \in \mathbb{N}\}$ is locally finite and:*
(a) *$|a_i| \subseteq U_i$ for any $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} a_i = 1$;*
(b) *$\sum_{i=1}^{\infty} C_{i,n} \cdot a_i$ is convergent for all $n \in \mathbb{N}$.*

Moreover, if $q_1 \leq q_2 \leq \ldots$ is a sequence of multiplicative semi-norms defining the topology of A, then such a sequence $(a_i)_{i \in \mathbb{N}}$ may be chosen so that the series $\sum_{i=1}^{\infty} q_n(a_i)$, $\sum_{i=1}^{\infty} C_{i,n} \cdot q_n(a_i)$ $(n \in \mathbb{N})$ are convergent.

DEFINITION 3.5. A ring is said to be *Gelfand* if each prime ideal is contained in a unique maximal ideal. This condition is equivalent to the existence, for any two maximal ideals $\mathfrak{m}_1 \neq \mathfrak{m}_2$, of elements $a \notin \mathfrak{m}_1$ and $b \notin \mathfrak{m}_2$ such that $a \cdot b = 0$.

THEOREM 3.6. *Let A be a rational Fréchet algebra. Then A is strictly regular if and only if it is regular and Gelfand.*

4. LOCALIZATION IN RATIONAL FRÉCHET ALGEBRAS

THEOREM 4.1. *Let A be a strictly regular rational Fréchet algebra. If U is a cozero-set in $\operatorname{Spec}_t A$, then the natural morphism $A_U \longrightarrow \tilde{A}(U)$ is an isomorphism.*

COROLLARY 4.2. *Let A be a strictly regular rational Fréchet algebra.*
(a) *$A = \tilde{A}(\operatorname{Spec}_t A)$.*
(b) *If A is separable, then the presheaf $U \mapsto A_U$ is a sheaf.*

A theorem of Brooks [3] follows from 4.2.(a): If A is a semisimple, regular and rational Fréchet algebra, then $A = \mathscr{A}(\operatorname{Spec}_t A)$ (see 2.1.(c)). Brooks also proved in [3] the existence of partitions of unity (see 3.4) for rational regular semisimple Fréchet algebras.

It is easy to prove that if A is a separable regular real Fréchet algebra, then every open set in $\operatorname{Spec}_t A$ is a cozero-set; 4.2.(b) follows from this result.

THEOREM 4.3. *Let A be a strictly regular rational Fréchet algebra. If U is a cozero-set in $\operatorname{Spec}_t A$, then A_U is a strictly regular rational Fréchet algebra with spectrum U.*

Idea of the proof. On the one hand, there exists on A_U a natural topology making it a strictly regular rational Fréchet algebra. On the other hand, for this topology the map $\pi : A \times S_U \longrightarrow A_U$ is continuous and has a continuous section. We conclude by 2.4 and 2.3. ∎

COROLLARY 4.4. *Let X be a Hausdorff, locally compact and σ-compact space. If U is a cozero-set in X, then the Fréchet algebra $\mathscr{C}(U)$ is the topological localization of $\mathscr{C}(X)$ with respect to $S = \{f \in \mathscr{C}(X) : 0 \notin f(U)\}$.*

COROLLARY 4.5. *Let V be a $\mathscr{C}^r$-manifold. If U is an open set in V, then the Fréchet algebra $\mathscr{C}^r(U)$ is the topological localization of $\mathscr{C}^r(V)$ with respect to $S = \{f \in \mathscr{C}^r(V) : 0 \notin f(U)\}$.*

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