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Classification of Symmetries for Higher Order Lagrangian Systems II: the Non-Autonomous Case*

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In this paper we continue the study of infinitesimal symmetries for higher order Lagrangian systems which we have iniciated in [3, 4]. A classification for first-order non-autonomous Lagrangian systems appears first time in Prince [6, 7] (see also Sarlet and Cantrijn [8] for higher order Cartan and Noether symmetries; Sarlet [9] for the relationship between equivalent Lagrangians and equivalence classes of dynamical symmetries; and Cariñena and Martínez [1] for a Noether theorem). In what concern to higher order Lagrangian systems, some results were obtained by Crampin, Sarlet and Cantrijn [2] (see also Sarlet [10, 11]).

Let Q be an n-dimensional manifold. The evolution space associated to Q is the manifold $J^k(\mathbb{R}, Q)$ of all k-jets, $j^k_t\sigma$, of mappings $\sigma : \mathbb{R} \longrightarrow Q$. We may canonically identify the manifolds $J^k(\mathbb{R}, Q)$ and $\mathbb{R} \times T^kQ$, where T^kQ is the tangent bundle of order k. We call $\beta^k : J^k(\mathbb{R}, Q) \longrightarrow Q$ and $\pi^k : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ the canonical projections defined, respectively, by $\beta^k(j^k_t\sigma) = \sigma(t)$ and $\pi^k(j^k_t\sigma) = t$. $J^k(\mathbb{R}, Q)$ is, also, a fibred bundle over $J^r(\mathbb{R}, Q)$, $0 \leq r \leq k$, being $\beta^k_r : J^k(\mathbb{R}, Q) \longrightarrow J^r(\mathbb{R}, Q)$ the projection defined by $\beta^k_r(j^k_t\sigma) = j^r_t\sigma$. Let $(t, q^A, q^A_1, \dots, q^A_k)$, $1 \leq A \leq n$, be the induced coordinates on $J^k(\mathbb{R}, Q)$ from local coordinates (q^A), $1 \leq A \leq n$, on Q, i.e., $q^A_i(j^k_t\sigma) = \frac{d^i}{dt^i}(q^A \circ \sigma(t))$, $1 \leq i \leq k$.

A C^∞-function $L : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ is said to be a non-autonomous (or time-dependent) Lagrangian of order k. We say that L is regular if the Hessian matrix $(\frac{\partial^2 L}{\partial q^A_k \partial q^B_k})$ is of maximal rank. We denote by E_L the energy associated to L and by Θ_L and $\Omega_L = -d\Theta_L$ the Poincaré-Cartan 1-form and 2-form, respectively. The intrinsic espressions of E_L and Θ_L are:

$$\begin{aligned} E_L &= \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}(\beta^{2k-1}_{k+r-1})^* \left(d_T^{r-1}(C_r L)\right) - (\beta^{2k-1}_k)^* L \, , \\ \Theta_L &= \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}(\beta^{2k-1}_{k+r-1})^* \left(d_T^{r-1}(d_{J_r} L)\right) - (\beta^{2k-1}_k)^* L \, dt \, . \end{aligned}$$

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where d_T is the total derivative respect to the time and $\bar{J}_r = J_r - C_r \otimes dt$, being $J_r = (J_1)^r$ and $C_r = J_{r-1}C_1$. Here J_1 denotes the canonical almost tangent structure of order k and C_1 the higher order Liouville vector field on T^kQ.

The global equations of the motion may be written as follows

$$i_X\Omega_L = 0 \ , \ i_X dt = 1 \ . \tag{1}$$

Since (Ω_L, dt) defines a cosymplectic structure on $J^{2k-1}(\mathbb{R}, Q)$, there exists a unique vector field ξ_L (the Reeb vector field) on $J^{2k-1}(\mathbb{R}, Q)$ satisfying (1). The vector field ξ_L will be called the Euler-Lagrange vector field. We have (see [5]):

1. ξ_L is a non-autonomous differential equation of order $2k$,
2. The solutions of ξ_L are just the solutions of the generalized Euler-Lagrange-equations:
$$\sum_{i=0}^{k}(-1)^i \frac{d^i}{dt^i}\left(\frac{\partial L}{\partial q_i^A}\right) = 0 \ .$$

In order to integrate the motion equations, it is useful to find functions which are constant along the motion. A differentiable function $f : J^{2k-1}(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ is called a **constant of the motion** of ξ_L if $\xi_L f = 0$.

We obtain a classification of infinitesimal symmetries in two classes, point symmetries (vector fields on $\mathbb{R} \times Q$) and not necessarily point-like symmetries (vector fields on $J^{2k-1}(\mathbb{R}, Q)$).

If X is a vector field on $\mathbb{R} \times Q$ we denote by $X^{(r,r)}$ the complete lift of X to $J^r(\mathbb{R}, Q)$ (see [12]). We obtain the following classification of point-symmetries:

Let X be a vector field on $\mathbb{R} \times Q$. Then:

1. X is said to be a **Lie symmetry** if
$$[\xi_L, X^{(2k-1,2k-1)}] = d_T(\tau)\xi_L \ ,$$
where $\tau = dt(X)$.
2. X is said to be a **Noether symmetry** if
$$L_{X^{(2k-1,2k-1)}}\Theta_L = df \ ,$$
for some function f on $J^{2k-1}(\mathbb{R}, Q)$.
3. X is said to be an **infinitesimal symmetry** of L if
$$X^{(k,k)}(L) = -d_T(\tau)L \ ,$$
where $\tau = dt(X)$.

We also obtain the following classification of not necessarily point symmetries: Let $\tilde{X}$ be a vector field on $J^{2k-1}(\mathbb{R}, Q)$. Then:

1. $\tilde{X}$ is said to be a **dynamical symmetry** of ξ_L if
$$[\xi_L, \tilde{X}] = \xi_L(\tau)\xi_L \ ,$$
where $\tau = dt(\tilde{X})$.

2. $\tilde{X}$ is called a **Cartan symmetry** if
$$L_{\tilde{X}}\Theta_L = df \ ,$$
for some function f on $J^{2k-1}(\mathbb{R}, Q)$.

We have obtained the following results which relate the different types of infinitesimal symmetries:

1. A Noether symmetry is a Lie symmetry.
2. An infinitesimal symmetry of L is a Noether symmetry.
3. A Cartan symmetry is a dynamical symmetry.
4. If X is a Noether symmetry then $X^{(2k-1,2k-1)}$ is a Cartan symmetry.
5. If X is a Lie symmetry then $X^{(2k-1,2k-1)}$ is a dynamical symmetry.

The relationship between symmetries and constants of the motion is given in the following results:

1. **(Noether theorem)** If $\tilde{X}$ is a Cartan symmetry of ξ_L then $F = f - \Theta_L(\tilde{X})$ is a constant of the motion of ξ_L. Conversely, if F is a constant of the motion of ξ_L then there exists a vector field Z on $J^{2k-1}(\mathbb{R}, Q)$ such that
$$i_Z\Omega_L = dF \ .$$
Hence, Z is a Cartan symmetry and every vector field $Z + g\xi_L$ with g: $J^{2k-1}(\mathbb{R}, Q) \to \mathbb{R}$ is also a Cartan symmetry.

2. F is a constant of the motion of ξ_L if and only if there exists a unique vector field $\tilde{X}$ on $J^{2k-1}(\mathbb{R}, Q)$ such that $dt(\tilde{X}) = 0$ and $L_{\tilde{X}}\Theta_L = d(\Theta_L(\tilde{X}) - F)$.

3. If X is an infinitesimal symmetry of L, then $\Theta_L(X^{(2k-1,2k-1)})$ is a constant of the motion of ξ_L.

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