EXTRACTA MATHEMATICAE Vol. 9, Núm. 2, 95 – 100 (1994)

A Lower Estimate of the Interface of some Nonlinear Diffusion Problems *

J. GONCERZEWICZ AND W. OKRASIŃSKI

Institute of Mathematics, University of Wroclaw, Pl. Grunwaldzki 2/4, 50-384 *Wroclaw, Poland*

AMS *Subject Class.* (1991): 35R35, 35K65, 35Q35, 76R50. *Received June* 10, 1994

1. INTRODUCTION

This paper presents some results concerning the behaviour of the interface of the following problem:

$$(1.1)\quad u_t = (u^m)_{xx} + (C/(x+a))(u^m)_x \qquad \text{for } (x,t) \in S = (0,\infty) \times (0,\infty)\,,$$

$$(1.2)\quad u(x,0) = u_0(x) \qquad \text{for } x \in (0,\infty)\,,$$

$$(1.3)\quad u(0,t) = u_1(t) \qquad \text{for } t \in (0,\infty)\,,$$

where $m > 1$, $C \geq 0$ and $a > 0$. We shall denote the above problem by $P(m, C; u_0, u_1)$. Throughout this paper we make the following assumptions:

$$(1.4)\quad \begin{array}{l} u_0 \in L^\infty(0,\infty),\ \text{ess inf}\, u_0 \geq 0,\ u_0 \equiv 0 \text{ a.e. on } (\alpha,\infty)\ (\alpha \geq 0)\,, \\ u_1 \in L^\infty(0,\infty),\ \text{ess inf}\, u_1 \geq \beta > 0\,. \end{array}$$

Without loss of generality we can assume that $\alpha = 1$ and $\beta = 1$.

In the case $C = 0$, equation (1.1) becomes the one-dimensional porous medium equation [2], [3], [10]. If $C = N - 1$ ($N = 2, 3, \dots$) then (1.1) is the radial version of the N-dimensional porous medium equation $u_t = \Delta(u^m)$, transformed by introducing the translated spatial variable [7]. Especially, the problem $P(2,1;0,1)$ describes the radially symmetrical infiltration into an unsaturated soil when the level of water in a cylindrical resorvoir is constant [9]. The question of interest is the range of infiltrating water.

Under assumptions (1.4), the problem $P(m, C; u_0, u_1)$ has a unique weak solution $u = u(x,t)$ [6], [7], [8]. The function u is nonnegative, bounded and continuous on S, and u satisfies an appropriate integral identity instead of (1.1). However, u is the classical solution for those points $(x,t) \in S$ where $u(x,t) > 0$. Moreover, if we define $\zeta(t) = \sup\{x \in (0,\infty) : u(x,t) > 0\}$ $(t > 0)$ then $0 < \zeta(t) < \infty$ for $t > 0$,

*This paper has been written when the second author was a Visiting Professor in the Universidad de Extremadura (Spain).

and $\zeta(t)$ is a Lipschitz continuous nondecreasing function. The curve $x = \zeta(t)$ is called the interface or the free boundary of $P(m, C; u_0, u_1)$.

We known that in the case of $P(m, 0; 0, 1)$ the interface has the form

$$\zeta(t) = c_0(m)t^{1/2}, \tag{1.5}$$

where the constant $c_0(m) > 0$ depends on m [1], [5], [11], [12], [16],[18], [19]. If $C \in [0, 1]$ then the interface of $P(m, C; u_0, u_1)$ satisfies the following asymptotic result [17]:

$$\log \zeta(t) \sim \frac{1}{2} \log t \qquad \text{as} \quad t \to \infty . \tag{1.6}$$

In this paper we construct a so-called weak subsolution of $P(m, C; u_0, u_1)$ for $C > 1$ and use this subsolution to prove tthe following theorem:

THEOREM. *Let $C > 1$. If ζ is the interface of the problem $P(m, C; u_0, u_1)$ then*

$$\zeta(t) \geq \left[(C - 1)(C + 1)m(m + 1)^{-1}t + 1\right]^{1/(C+1)} - 1 , \qquad t \geq 0 . \tag{1.7}$$

In the authors' opinion, the estimate (1.7) seems to be useful for futher considerations concerning the large-time behaviour of ζ.

2. Some Information about Weak Subsolutions

We recall some results presented in [7]. We put

$$\mathscr{L}(u) \equiv (u^m)_{xx} + (C/(x + 1))(u^m)_x - u_t .$$

The following facts concerning weak subsolutions shall be needed:

LEMMA 2.1. *Let $\gamma \in \mathscr{C}([0, \infty)) \cap \mathscr{C}^1((0, \infty))$, $\gamma(t) \geq 0$ for $t \geq 0$ and let $\Gamma = \{(x, t) : x = \gamma(t), t \geq 0\}$. If:*

(i) *$\underline{u}$ is nonnegative bounded and continuous on $S = (0, \infty) \times (0, \infty)$,*

(ii) $\underline{u}_t, (\underline{u}^m)_{xx} \in \mathscr{C}(S \backslash \Gamma)$, $(\underline{u}^m)_x \in \mathscr{C}(S)$,

(iii) *$\underline{u}(x, 0+)$ exists for a.e. $x \in [0, \infty)$ and $\underline{u}(0+, t)$ exists for a.e. $t \in [0, \infty)$,*

(iv) *$\mathscr{L}(\underline{u}) \geq 0$ in $S \backslash \Gamma$,*

then $\underline{u}$ is a weak subsolution of $P(m, C; \underline{u}_0, \underline{u}_1)$ with $\underline{u}_0 = \underline{u}(\cdot, 0+)$ and $\underline{u}_1 = (0+, \cdot)$.

LEMMA 2.2. *Let u be the weak solution of $P(m, C; u_0, u_1)$ and let $\underline{u}$ be a weak subsolution of $P(m, C; \underline{u}_0, \underline{u}_1)$. If $\underline{u}_0 \leq u_0$ a.e. and $\underline{u}_1 \leq u_1$ a.e., then $\underline{u} \leq u$ a.e. on S.*

In the next two sections we shall construct a weak subsolution of $P(m, C; 0, 1)$ for $C > 1$.

3. An Auxilliary Differential Equation

Let $m > 1$ and $C > 1$. We consider the ordinary differential equation

$$(f^m)'' + (C/s)(f^m)' = -\frac{1}{2}sf', \quad s \in (0,1], \tag{3.1}$$

with conditions

$$f(1) = 0, \quad \lim_{s\to 1-}(f^m(s))' = 0. \tag{3.2}$$

We shall look for nonnegative nontrivial solutions $f \in \mathscr{C}((0,1]) \cap \mathscr{C}^2((0,1))$ of (3.1)–(3.2). Using the substitution

$$f(s) = g(s^{1-C} - 1), \quad s \in (0,1], \tag{3.3}$$

the above problem is transformed into

$$(g^m)'' = \big(1/[2(C-1)]\big)(x+1)^{(C+1)/(1-C)}g', \quad x \in [0,\infty), \tag{3.4}$$

$$g(0) = 0, \quad \lim_{x\to 0+}(g^m(x))' = 0. \tag{3.5}$$

The following lemma is the key point in our considerations:

Lemma 3.1. *The problem* (3.4)–(3.5) *has the unique solution $g \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^2((0,\infty))$ such that $g(x) > 0$ for $x > 0$. Moreover g is strictly increasing and*

$$g(x) \le \big[((m-1)/(2m(C-1))x\big]^{1/(m-1)}, \quad x \ge 0. \tag{3.6}$$

Sketch of proof. We follow the ideas of [13] and [14]. Using the substitution (see [15])

$$v(x) = g^m(x),$$

the problem (3.4)–(3.5) can be reduced to the nonlinear integral Volterra equation

$$v(x) = \int_0^x k(x,s)[v(s)]^{1/m}\,\mathrm{d}s, \quad x \in [0,\infty), \tag{3.7}$$

where

$$\begin{aligned} k(x,s) \equiv \big(1/[2(C-1)]\big)\Big(&(s+1)^{(C+1)/(1-C)} \\ &+ \big[(C+1)/(C-1)\big](x-s)(s+1)^{2C/(1-C)}\Big), \quad 0 \le s \le x. \end{aligned} \tag{3.8}$$

In view of the results of [4] and [14], there exists a unique nontrivial solution of (3.7), i.e., a continuous function v such that $v(x) > 0$ for $x > 0$ ($v \equiv 0$ is the trivial solution of (3.7)). This implies that the first part of the lemma is true. Differentiating (3.7) we obtain

$$v'(x) = k(x,x)[v(x)]^{1/m} + \int_0^x k_x(x,s)[v(s)]^{1/m}\,\mathrm{d}s, \quad x \in (0,\infty). \tag{3.9}$$

Since $k_x(x,s) > 0$ for $x > 0$ and $0 \le s \le x$, then $v'(x) > 0$ for $x > 0$. Hence, v (and, consequently, g) is strictly increasing. Moreover, from (3.9), we get the following inequality

$$v'(x) \le \big(1/[2(C-1)]\big)[v(x)]^{1/m}, \quad x \in (0,\infty).$$

Integration gives

$$(m/(m-1))[v(x)]^{(m-1)/m} \le \big(1/[2(C-1)]\big)x, \quad x \in (0,\infty).$$

Since this last inequality is equivalent to (3.6), the lemma is proved. ∎

An inmediate consequence of Lemma 3.1 is the following.

COROLLARY 3.1. *The problem* (3.1)–(3.2) *has a unique solution* $f \in \mathscr{C}((0,1]) \cap \mathscr{C}^2((0,1))$ *such that* $f(s) > 0$ *for* $s \in (0,1)$. *Moreover* f *is strictly decreasing and*

$$f(s) \le \Big[\big((m-1)/[2m(C-1)]\big)(s^{1-C}-1)\Big]^{1/(m-1)}, \quad s \in (0,1]. \tag{3.10}$$

4. CONSTRUCTION OF A SUBSOLUTION

We start with the following problem:

$$\dot{A} = 1/[f(1/A^{1/2})]^{m-1}, \tag{4.1}$$

$$A(0) = 1, \tag{4.2}$$

where f is the function whose existence is asserted in Corollary 3.1.

The following lemma holds.

LEMMA 4.1. *The problem* (4.1)–(4.2) *has a unique solution* $A \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^2((0,\infty))$, *which is a strictly increasing function.*

Since the proof of Lemma 4.1 is very similar to the proof of Lemma 5.1 in [13] we omit its details.

Since f is decreasing and A is increasing, then, by (4.1), A is concave and, consequently,

$$\ddot{A}(t) \le 0, \quad t > 0. \tag{4.3}$$

Now, we define a function $\underline{u}$ by

$$\underline{u}(x,t) = \begin{cases} [\dot{A}(t)]^{1/(m-1)} f((x+1)/[A(t)]^{1/2}) & , \quad \text{for } x \le [A(t)]^{1/2} - 1, \\ 0 & , \quad \text{for } x > [A(t)]^{1/2} - 1. \end{cases} \tag{4.4}$$

LEMMA 4.2. *If* $C > 1$ *then the function* $\underline{u}$ *defined by* (4.4) *is a weak subsolutiom of* $P(m,C;0,1)$.

Proof. Let $\gamma(t) = [A(t)]^{1/2} - 1$. By Corollary 3.1 and (4.4) it is easy to see that $\underline{u}$ satisfies assumptions (i)–(iii) of the Lemma 2.1 with $\underline{u}(x, 0+) = 0$ for $x \in (0, \infty)$ and $\underline{u}(0+, t) = 1$ for $t \in (0, \infty)$. If $t > 0$ and $x \in (0, \gamma(t))$, then we have

$$\mathscr{L}(\underline{u})(x,t) = (-1/(m-1))\, \ddot{A}(t)\, [\dot{A}(t)]^{(2-m)/(m-1)} f\big((x+1)/[A(t)]^{1/2}\big)\,.$$

By (4.3) and (4.4) we infer $\mathscr{L}(\underline{u}) \geq 0$ in $S \backslash \Gamma$. Thus, the assumption (iv) of Lemma 2.1 is fulfiled. ∎

5. Proof of Theorem

We compare the weak solution u of $P(m, C; u_0, u_1)$ with the weak subsolution $\underline{u}$ defined in the previous section. Since $\underline{u}_0 \leq u_0$ a.e. in $(0, \infty)$ and $\underline{u}_1 \leq u_1$ a.e. in $(0, \infty)$ then, according to Lemma 2.2, $\underline{u} \leq u$ on S. Hence

$$\zeta(t) \geq [A(t)]^{1/2} - 1\,, \quad t \geq 0\,. \tag{5.1}$$

It follows from (3.10) and (4.1)–(4.2) that

$$A(t) \geq w\big((2(C-1)m/(m-1))t\big)\,, \quad t \geq 0\,, \tag{5.2}$$

where $w = w(\vartheta)$ is defined by

$$\vartheta = \int_1^{w(\vartheta)} (\xi^{(C-1)/2} - 1)\, \mathrm{d}\xi\,, \quad \vartheta > 0\,.$$

It is easy to see that

$$w(\vartheta) \geq \Big[((C+1)/2)\vartheta + 1\Big]^{2/(C+1)}\,, \quad \vartheta \geq 0\,. \tag{5.3}$$

Combining (5.1), (5.2) and (5.3) we obtain (1.7). ∎

References

[1] Atkinson, F.V. and Peletier, L.A., Similarity profiles of flows through porous media, *Arch. Rat. Mech. Anal.* **42** (1971), 369–379.

[2] Bear, J., "Dynamics of Fluids in Porous Media", Elsevier, New York, 1971.

[3] Bertsch, M. and Peletier, L.A., "Porous Media Type Equation: an Overview" Preprint No. 7, Mathematical Institute, University of Leiden, 1983.

[4] Bushell, P.J., On a class of Volterra and Fredholm nonlinear integral equations, *Math. Proc. Camb. Phil. Soc.* **79** (1976), 329–335.

[5] Gilding, B.H. and Peletier, L.A., On a class of similarity solutions of the porous media equation II, *J. Math. Anal. Appl.* **57** (1977), 522–538.

[6] Goncerzewicz, J., Porous medium-type equation with irregular boundary data, in "Proc. Int. Conf. on Recent Advances in Non-linear Eliptic and Parabolic Problems, Nancy, France, March 1988", Eds. Ph. Bénilan et al., Pitman Res. Notes in Math. Sci. Series 208, Longman, London 1989, 59–67.

[7] Goncerzewicz, J., Properties of solutions to the initial-boundary value problem for a porous media-type equation, *Math. Meth. in the Appl. Sci.* **15** (1992), 299–314.

[8] GONCERZEWICZ, J. , On an initial-boundary value problem for a certain class of degenerate parabolic equations, *in preparation.*

[9] GONCERZEWICZ, J., MARCINKOWSKA, H., OKRASIŇSKI, W. AND TABISZ, K. , On the percolation of water from a cylindrical reservoir into the sorrounding soil, *Zastosow. Mat.* **16** (1978), 249–261.

[10] KALASHNIKOV, A.S. , Some properties of the qualitative theory of second order nonlinear degenerate parabolic equations, *Usp. Mat. Nauk* **42**(2) (1987), 135–176.

[11] KING, J.R. , High concentration arsenic diffusion in crystalline silicon – an asymptotic analysis, *IMA J. Appl. Mat.* **38** (1987), 87–95.

[12] KING, J.R. , Approximate solutions to a nonlinear diffusion equation, *J. Engrg. Math.* **22** (1988), 53–72.

[13] OKRASIŇSKI, W. , On subsolutions of a nonlinear diffusion problem, *Math. Meth. in the Appl. Sci.* **11** (1989), 409–416.

[14] OKRASIŇSKI, W. , On a nonlinear ordinary differential equation, *Annal. Polon. Math.* **49** (1989), 237–245.

[15] OKRASIŇSKI, W. , Nontrivial solutions to nonlinear Volterra integral equations, *SIAM J. Math. Anal.* **22**(4) (1991), 1007–1015.

[16] OKRASIŇSKI, W. , On approximate solutions to some nonlinear diffusion problems, *Z. Angew. Math. Phys.* **44** (1993), 722–731.

[17] OKRASIŇSKI, W. , Asymptotic behaviour of the free boundary of some nonlinear diffusion problems, (submitted for publication).

[18] OKRASIŇSKI, W. AND VILA, S. , Power series solutions to some nonlinear diffusion problems, *Z. Angew. Math. Phys.* **44** (1993), 988–997.

[19] RYBARSKI, A. , The integral equation describing propagation of the ground water tongue, preprint in Polish, 1973.