EXTRACTA MATHEMATICAE Vol. 9, Núm. 2, 88–94 (1994)

Some Remarks on Hadamard's Inequalities for Convex Functions

SEVER SILVESTRU DRAGOMIR

Dep. of Math., University of Timişoara, B-dul V. Pârvan 4, R-1900 *Timişoara, Románia*

(Presented by Carlos Benítez)

AMS *Subject Class.* (1991): 26D15. *Received October* 7, 1993

1. Introduction

In paper [4] we introduced the following two mappings associated to a convex function $f\colon [a,b] \to \mathbb{R}$; $H, F\colon [0,1] \to \mathbb{R}$ given by

$$H(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\,dx$$

and

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dx\,dy$$

and we proved the following main properties

(i) H, F are convex in $[0,1]$.

(ii) H increases monotonically on $[0,1]$, F is decreasing on $[0,1/2]$ and increasing on $[1/2,1]$.

(iii) We have the bounds

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right) ;$$

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx ;$$

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx ;$$

$$\inf_{t\in[0,1]} F(t) = F(1/2) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\,dx\,dy .$$

(iv) One has the inequalities

$$f\left(\frac{a+b}{2}\right) \le F(1/2) \quad \text{and} \quad H(t) \le F(t) \quad \text{for all } t\in[0,1] .$$

The main aim of this note is to give another type of refinements to the classical inequality due to Hadamard

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} .$$

For other inequalities connected with this main result in Mathematical Analysis, we send to the recent papers [1–10] where further references are given.

2. The Main Results

Let $[a,b]$ be a compact interval of real numbers, $d := \{x_i \mid i = \overline{0,n}\} \subset [a,b]$ a division of the interval $[a,b]$ given by

$$d \; : \; a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \qquad (n \ge 1)$$

and f a bounded mapping on $[a,b]$. We consider the following sums

$$h_d(f) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) \quad \text{(called Hadamard's inferior sum)}$$

$$H_d(f) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \quad \text{(called Hadamard's superior sum)}$$

and Darboux's sums

$$s_d(f) := \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) , \qquad S_d(f) := \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i)$$

where

$$m_i = \inf_{x\in[x_i,x_{i+1}]} f(x) \quad , \quad M_i = \inf_{x\in[x_i,x_{i+1}]} f(x) \quad , \quad i = 0,\ldots,n-1 .$$

It is well-known that f is Riemann integrable on $[a,b]$ if and only if

$$\sup_d s_d(f) = \inf_d S_d(f) = I \in \mathbb{R}$$

and in this case

$$I = \int_a^b f(x)\,dx .$$

The following theorem holds:

THEOREM. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then*

(i) *$h_d(f)$ increases monotonically over d, i.e. for $d_1 \subseteq d_2$ one has $h_{d_1}(f) \le h_{d_2}(f)$.*

(ii) *$H_d(f)$ is decreasing over d.*

(iii) *We have the bounds*

$$\frac{1}{b-a}\inf_d h_d(f) = f\left(\frac{a+b}{2}\right), \quad \sup_d h_d(f) = \int_a^b f(x)\,dx \tag{1}$$

and

$$\inf_d H_d(f) = \int_a^b f(x)\,dx, \quad \sup_d H_d(f) = \frac{f(a)+f(b)}{2}\,. \tag{2}$$

Proof. (i) Without lost of generality we can assume that $d_1 \subseteq d_2$ with $d_1 = \{x_0, \ldots, x_n\}$ and $d_2 = \{x_0, \ldots, x_k, y, x_{k+1}, \ldots x_n\}$ where $y \in [x_k, x_{k+1}]$ $(0 \le k \le n-1)$. Then

$$h_{d_2}(f) - h_{d_1}(f) = f\left(\frac{x_k+y}{2}\right)(y-x_k) + f\left(\frac{y+x_{k+1}}{2}\right)(x_{k+1}-y) - f\left(\frac{x_k+x_{k+1}}{2}\right)(x_{k+1}-x_k).$$

Let put

$$\alpha = \frac{y-x_k}{x_{k+1}-x_k}, \quad \beta = \frac{x_{k+1}-y}{x_{k+1}-x_k}, \quad x = \frac{x_k+y}{2}, \quad z = \frac{y+x_{k+1}}{2}\,.$$

Then

$$\alpha+\beta = 1, \qquad \alpha x + \beta y = \frac{x_k+x_{k+1}}{2}$$

and by the convexity of f we deduce that $\alpha f(x)+\beta f(z) \ge f(\alpha x+\beta z)$, i.e. $h_{d_2}(f) \ge h_{d_1}(f)$.

(ii) For d_1, d_2 as above, we have

$$\begin{aligned} H_{d_2}(f) - H_{d_1}(f) &= \frac{f(x_k)+f(y)}{2}(y-x_k) \\ &\quad + \frac{f(y)+f(x_{k+1})}{2}(x_{k+1}-y) + \frac{f(x_k)+f(x_{k+1})}{2}(x_{k+1}-x_k) \\ &= \frac{f(y)(x_{k+1}-x_k)}{2} - \frac{f(x_k)(x_{k+1}-y)+f(x_{k+1})(y-x_k)}{2}\,. \end{aligned}$$

Now, let α, β be as above and $u = x_k$, $v = x_{k+1}$. Then $\alpha u + \beta v = y$ and by the convexity of f we have $\alpha f(u) + \beta f(v) \ge f(y)$, i.e. $H_{d_2}(f) \le H_{d_1}(f)$ and the statement is proved.

(iii) Let $d = \{x_0, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$. Put $p_i := x_{i+1} - x_i$, $u_i = (x_{i+1}+x_i)/2$, $i = 0, \ldots, n-1$. Then by Jensen's discrete inequality

$$f\left(\frac{\sum_{i=0}^{n-1} p_i u_i}{\sum_{i=0}^{n-1} p_i}\right) \le \frac{\sum_{i=0}^{n-1} p_i f(u_i)}{\sum_{i=0}^{n-1} p_i}$$

and since

$$\sum_{i=0}^{n-1} p_i = b - a, \quad \sum_{i=0}^{n-1} p_i u_i = \frac{b^2 - a^2}{2},$$

we deduce the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} h_d(f).$$

If $d = d_0 = \{a, b\}$, we obtain $h_{d_0}(f) = f\big((a+b)/2\big)$, which proves the first bound in (1).

By the first inequality in Hadamard's result, we have

$$f\left(\frac{x_i + x_{i+1}}{2}\right) \le \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)\,dx, \quad i = 0, \dots, n-1,$$

which gives, by addition,

$$\begin{aligned} h_d(f) &= \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) \\ &\le \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \\ &= \int_a^b f(x)\,dx, \end{aligned}$$

for all d a division of $[a, b]$.

Since

$$s_d(f) \le h_d(f) \le \int_a^b f(x)\,dx, \quad d \text{ is a division of } [a, b],$$

and f is Riemann integrable on $[a, b]$, i.e.

$$\sup_d s_d(f) = \int_a^b f(x)\,dx,$$

it follows that

$$\sup_d h_d(f) = \int_a^b f(x)\,dx,$$

which proves the relation (1).

To prove the relation (2), we observe, by the second inequality in Hadamard's result, that

$$\begin{aligned} \int_a^b f(x)\,dx &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx, \\ &\le \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \\ &= H_d(f) \end{aligned}$$

where d is an arbitrary division of $[a, b]$.

Since

$$H_d(f) \leq S_d(f), \quad \text{for all } d \text{ as above,}$$

and f is integrable on $[a, b]$, we conclude that

$$\inf_d H_d(f) = \int_a^b f(x)\,dx$$

Finally, because for all d a division of $[a, b]$ we have $d \supseteq d_0 = \{a, b\}$, thus

$$\sup_d H_d(f) = \frac{f(a) + f(b)}{2}$$

and the theorem is proved. ∎

Remark. Let f be a convex mapping on $[a, b]$. Then for all $a = x_0 < x_1 < \cdots < x_n = b$, we have the following improvement of Hadamard's result

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) \\ &\leq \frac{1}{b-a} \int_a^b f(x)\,dx \\ &\leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{3}$$

COROLLARY 1. *Let f be as above. Define the sequences*

$$h_n(f) := \frac{1}{n} \sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right)$$

$$H_n(f) := \frac{1}{2n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{i}{n}(b-a)\right) + f\left(a + \frac{i+1}{n}(b-a)\right)\right]$$

for $n \geq 1$. Then we have the inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h_n(f) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \\ &\leq H_n(f) \leq \frac{f(a) + f(b)}{2}, \qquad n \geq 1. \end{aligned} \tag{4}$$

Moreover, one has

$$\lim_{n \to \infty} h_n(f) = \lim_{n \to \infty} H_n(f) = \frac{1}{b-a} \int_a^b f(x)\,dx. \tag{5}$$

Proof. The inequalities (4) follows by (3) for $d := \{x_i = a + \frac{i}{n}(b-a) \mid i = \overline{0,n}\}$. The relation (5) is obvious by the integrability of f. We omit the details. ∎

COROLLARY 2. *Let $f\colon [a,b] \to \mathbb{R}$ be a convex mapping on $[a,b]$. Define the sequences*

$$t_n(f) := \frac{1}{2^n}\sum_{i=0}^{n-1} f\left(a + \frac{2^i}{2^{n+1}}(b-a)\right)2^i$$

and

$$T_n(f) := \frac{1}{2^{n+1}}\sum_{i=0}^{n-1}\left[f\left(a + \frac{2^i}{2^n}(b-a)\right) + f\left(a + \frac{2^{i+1}}{2^n}(b-a)\right)\right]2^i$$

($n \geq 1$). Then we have

(i) *t_n is monotonous increasing;*

(ii) *T_n is monotonous decreasing;*

(iii) *The following identities are valid*

$$\sup_{n\geq 1} t_n(f) = \lim_{n\to\infty} t_n(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\inf_{n\geq 1} T_n(f) = \lim_{n\to\infty} T_n(f) = \frac{1}{b-a}\int_a^b f(x)\,dx\ .$$

Proof. (i), (ii). Is obvious by (i) and (ii) of Theorem for

$$d_n := \left\{x_i = a + \frac{2^i}{2^n}(b-a) \mid i = \overline{0,n}\right\} \subseteq d_{n+1}, \quad n \in \mathbb{N}.$$

(iii). It follows from bounds (1), (2) and the fact that f is Riemann integrable on $[a,b]$.

APPLICATIONS. a) Let $0 \leq a = x_0 < x_1 < \cdots < x_n = b$ and $p \geq 1$. Then we have the inequalities

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{b-a}\sum_{i=0}^{n-1}\left(\frac{x_i + x_{i+1}}{2}\right)^p (x_{i+1} - x_i) \leq \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}$$
$$\leq \frac{1}{b-a}\sum_{i=0}^{n-1}\frac{x_i^p + x_{i+1}^p}{2}(x_{i+1} - x_i) \leq \frac{a^p + b^p}{2}.$$

b) Suppose that $0 < a$ and x_i are as above. Then one has

$$\frac{2}{a+b} \leq \frac{2}{b-a}\sum_{i=0}^{n-1}\frac{(x_{i+1} - x_i)}{(x_{i+1} + x_i)} \leq \frac{\ln b - \ln a}{b-a}$$
$$\leq \frac{1}{b-a}\sum_{i=0}^{n-1}\frac{x_{i+1}^2 - x_i^2}{2x_i x_{i+1}} \leq \frac{a+b}{2ab}\ .$$

c) We have the following refinement of arithmetic mean-geometric mean inequality

$$\begin{aligned}\frac{a+b}{2} &\geq \prod_{i=0}^{n-1}\left(\frac{x_i+x_{i+1}}{2}\right)^{\frac{(x_{i+1}-x_i)}{b-a}} \geq \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \\ &\geq \prod_{i=0}^{n-1}(x_i x_{i+1})^{\frac{(x_{i+1}-x_i)}{2(b-a)}} \geq \sqrt{ab}\end{aligned}$$

where $a > 0$ and x_i are as above.

References

[1] ALZER, H., A note on Hadamard's inequalities, *C.R. Math. Rep. Acad. Sci. Canada,* **11** (1989), 255–258.

[2] DRAGOMIR, S.S., Some refinements of Hadamard's inequalities, *C.M. Metod (Bucharest),* **11** (1990), 189–191.

[3] DRAGOMIR, S.S., Two refinements of Hadamard's inequalities, *Coll. Sci. Pap. Fac. Sci. Kragujevać,* **11** (1990), 23–26.

[4] DRAGOMIR, S.S., Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* (1992) (in press) .

[5] DRAGOMIR, S.S., PEČARIĆ, J.E., SÁNDOR, J., A note on the Jensen–Hadamard inequalities, *Anal. Numer. Theor. Approx.,* **19** (1990), 29–34.

[6] DRAGOMIR, S.S., PEČARIĆ, J.E., A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.,* **7** (1991), 103–107.

[7] PEČARIĆ, J.E., A note on Hadamard's inequalities, *Macedonian Acad. Sci. and Arts, Sec. Math. Sci.,* **10** (1989), 21–23.

[8] SÁNDOR, J., Some integral inequalities, *Elem. Math.,* **43** (1988), 177–180.

[9] SÁNDOR, J., An application of the Jensen–Hadamard inequality, *Nieuw. Arch. Wiskunde,* (in press) .

[10] VASIĆ, P.M., LACKOVIĆ, I.B., Notes on convex functions (1), *Univ. Beograd Publ. Elektr. Fak. Ser. Mat. Fiz.,* No. **577**– No. **598** (1977), 21–24.