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A Solution to the $\bar{\partial}$–Problem for Holomorphic $(0,q)$–Forms, $q \geqslant 1$, on a Complex Normed Space

ROBERTO LUIZ SORAGGI[1]

Dpto de Matem. Pura, Universidade Federal do Rio de Janeiro,
C.P. 68530, *CEP* 21944, *Rio de Janeiro, Brazil*

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1. INTRODUCTION

Using entirely elementary methods from differential calculus, we construct a $\mathscr{C}^\infty$– solution to the equation $\bar{\partial}u = \omega$ where ω is a holomorphic $(0,q)$ form on a normed space or a Fréchet–Montel space or a DFM space. This extends in certain directions results in [4], [6] and [7]. This is in sharp constrast with the situation for the Cauchy–Riemann equation for $\mathscr{C}^\infty(0,q)$ forms ω where satisfactory solutions are only known for $q = 1$ on DFN spaces. Counterexamples in [1], [3] and [5] show that restrictions on the spaces, the coefficients and the degree are necessary. For $q > 1$, solutions are given for coefficients with polynomial growth in ℓ_2 [4] and this is the only known result for arbitrary q. Solutions for $q = 1$ on separable Hilbert spaces and DFN spaces are given in [6]. The case $q = 2$ is solved with holomorphic coefficients in [7] and in [9] solutions are given for $(0,1)$ holomorphic forms on a Fréchet nuclear space.

2. THE CONSTRUCTION OF A SOLUTION OF $\bar{\partial}u = \omega$ FOR HOLOMORPHIC $(0,q)$ FORMS, $q \geqslant 1$, ON A NORMED SPACE

Let E and F be complex normed spaces. Let $\mathbb{K}$ be the field of real or complex numbers. For a positive integer $q \geqslant 1$, $\mathscr{L}_{\mathbb{K}}({}^q E; F)$ is the normed vector space of continuous q–$\mathbb{K}$–linear mappings from E into F. Let $\mathscr{L}_{\mathbb{K}}({}^o E; F) = F$. The conjugate space of E will be denoted by $\overline{E}$. Let $\mathscr{L}_{\bar{\mathbb{C}}}({}^q E; F)$ be the normed vector space of continuous q– anti–linear mappings from E into F. The notations $\mathscr{L}_{\bar{\mathbb{C}}}({}^q E; F) = \mathscr{L}_{\mathbb{C}}({}^q \overline{E}; F) = \mathscr{L}({}^q \overline{E}; F)$ will be used.

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We denote by $\mathscr{L}_{\mathbb{K}}({}^qE)$ the Banach space of continuous $q-\mathbb{K}-$linear forms on E. The Banach space of continuous $q-$anti–linear forms on E will be denoted by $\mathscr{L}_{\bar{\mathbb{C}}}({}^qE)$. For $q \geqslant 1$, let $\Lambda^{(0,q)}(E)$ be the Banach space of continuous alternating forms on E. For $q=0$ let $\Lambda^{(0,q)}(E) = \mathscr{L}_{\bar{\mathbb{C}}}({}^qE) = \mathbb{C}$.

If $\omega : \Omega \longrightarrow \Lambda^{(0,q)}(E)$ is a $\mathscr{C}^1(0,q)$ form on an open subset Ω of E let $\omega' : \Omega \longrightarrow \mathscr{L}_{\mathbb{R}}(E, \Lambda^{(0,q)}(E))$ be its derivative. Let $[\bar{\partial}]\omega(z) \in \mathscr{L}(\bar{E}, \Lambda^{(0,q)}(E))$ be the anti–linear component of $\omega'(z)$ and let $\bar{\partial}\omega(z)$ be the alternating component of $[\bar{\partial}]\omega(z)$. We use without explicit mention the isometry between $\mathscr{L}(E, \mathscr{L}({}^qE;F), \mathscr{L}({}^{q+1}E;F))$ and $\mathscr{L}({}^2E, \mathscr{L}({}^{q-1}E;F))$, $q \geqslant 1$. For each fixed $z \in E$ and $q > 1$ define $\tau_q(z) : E^{q-1} \longrightarrow E^q$ by $\tau_q(z)(h_1, \ldots, h_{q-1}) = (h_1, \ldots, h_{q-1}, z)$. For $q=1$, $\tau_1(z) = z$. For further details we refer to [7].

We now state some remarks which we use in the proof of our main result. The proofs are straightforward.

Remark 2.1. Let $T \in \Lambda^{(0,q)}(E) \hookrightarrow \mathscr{L}({}^q\bar{E})$, $q \geqslant 2$. For each fixed $z \in E$, the function $T_1 : (\bar{E})^{q-1} \longrightarrow \mathbb{C}$ defined by $T_1(z_1, \ldots, z_{q-1}) = T_1(z, z_1, \ldots, z_{q-1})$ is a continuous alternating $(q-1)$ anti–linear form on E. We write $T_1 = T(z)$. In particular, for a $(0,q)$ form ω on E, we have that $u : E \longrightarrow \Lambda^{(0,q-1)}(E)$ defined by $u(z) = \omega(z)(z)$ is a $(0, q-1)$ form on E.

Remark 2.2. Let $\omega : \Omega \longrightarrow \mathscr{L}_{\mathbb{R}}({}^qE;F)$, $q \geqslant 1$, be a $\mathscr{C}^1$ mapping. For z in Ω and h in E, $\lim_{t\to 0}(\omega(z+th) - \omega(z))/t = \omega'(z)h \in \mathscr{L}_{\mathbb{R}}({}^qE;F)$. Continuity of ω at z implies that $\omega(z+th)$ tends to $\omega(z)$ in $\mathscr{L}_{\mathbb{R}}({}^qE;F)$ as $t \longrightarrow 0$. For each fixed $z \in E$ the mapping $T \in \mathscr{L}_{\mathbb{R}}({}^qE;F) \longrightarrow T \circ \tau_q(z) \in \mathscr{L}({}^{q-1}E;F)$ is continuous. Hence it follows easily that $A : \Omega \longrightarrow \mathscr{L}_{\mathbb{R}}({}^{q-1}E;F)$ defined by

$$A(z)(h_1, \ldots, h_{q-1}) = \omega(z) \circ \tau_q(z)(h_1, \ldots, h_{q-1}, z) = \omega(z)(h_1, \ldots, h_{q-1}, z)$$

is $\mathscr{C}^1$ and

$$A'(z) = \omega'(z) \circ \tau_{q+1}(z) + \omega(z) \circ \tau_q \tag{1}$$

Remark 2.3. Let $\omega : \Omega \longrightarrow \mathscr{L}_{\mathbb{R}}({}^qE;F)$, $q \geqslant 3$, be a $\mathscr{C}^1$ mapping. For all $k = 0, 1, \ldots, q-2$, the mapping $B_k : \Omega \longrightarrow \mathscr{L}_{\mathbb{R}}({}^qE;F)$ defined by

$$\begin{aligned} B_k(z)(h_1, \ldots, h_q) &= \omega(z)(h_1, \ldots, h_k) \circ \tau_{q-k}(h_{k+1})(h_{k+2}, \ldots, h_q) \\ &= \omega(z)(h_1, \ldots, h_k, h_{k+2}, \ldots, h_q, h_{k+1}) \end{aligned}$$

is $\mathscr{C}^1$ and

$$\begin{aligned} \mathrm{B}_k'(z)(h)(h_1,\dots,h_q) &= [\omega'(z).h](h_1,\dots,h_k,h_{k+2},\dots,h_q,h_{k+1}) \\ &= \omega'(z)(h,h_1,\dots,h_k)\circ\tau_{q-k}(h_{k+1})(h_{k+2},\dots,h_q). \end{aligned} \tag{2}$$

Now, we have the following crucial lemma.

LEMMA 2.4. *Let* $\omega:\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(^2E;F)$ *be a* $\mathscr{C}^\infty$ *mapping. Define* $A:\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(E;F)$ *by* $A(z)=\omega(z)\circ\tau_2(z)$. *Then* A *is a* $\mathscr{C}^\infty$ *mapping and for* $n\geqslant 1$

$$A^{(n)}(z)=\omega^{(n)}(z)\circ\tau_{n+2}(z)+\sum_{k=0}^{n-1}B_k(z). \tag{3}$$

Where $B_k(z)(h_1,\dots,h_{n+1})=\omega^{(n-1)}(z)(h_1,\dots,h_k)\circ\tau_{n+1-k}(h_{k+1})(h_{k+2},\dots,h_{n+1})$.

Proof. We prove this result by induction. Let $n=1$. $A':\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(E,\mathscr{L}_{\mathbf{R}}(E;F))\cong\mathscr{L}_{\mathbf{R}}(^2E;F)$. By applying (1) (in remark 2.2) we have

$$\begin{aligned} A^{(n)}(z) &= \omega'(z)\circ\tau_{2+1}(z)+\omega(z)\circ\tau_2=\omega'(z)\circ\tau_{1+2}(z)+\omega(z)\circ\tau_{1+1} \\ &= \omega^{(n)}(z)\circ\tau_{n+2}(z)+\omega^{(1-1)}(z)\circ\tau_{n+1} \\ &= \omega^{(n)}(z)\circ\tau_{n+2}(z)+B_0(z). \end{aligned}$$

Suppose (3) is true for $n\geqslant 1$.

We consider $B:\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(^{n+1}E;F)$ defined by $B(z)=\omega^{(n)}(z)\circ\tau_{n+2}(z)$. Since $\omega^{(n)}:E\longrightarrow\mathscr{L}(^{n+2}E;F)$ is $\mathscr{C}^1$, B is $\mathscr{C}^1$ and

$$B'(z)=\omega^{(n+1)}(z)\circ\tau_{n+2+1}(z)+\omega^{(n)}(z)\circ\tau_{n+2}. \tag{4}$$

On the other hand the mapping $B_k:\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(^{n-1}E;F)$ for $k=0,1,\dots,n-1$ is $\mathscr{C}^1$ and $B_k':\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(^{n+2}E;F)$ is given by

$$\begin{aligned} &B_k'(z)(h)(h_1,\dots,h_{n+1}) \\ &= \omega^{(n)}(z)(h)(h_1,\dots,h_k)\circ\tau_{n+1-k}(h_{k+1})(h_{k+2},\dots,h_{n+1}) \\ &= \omega^{(n)}(z)(h)(h_1,\dots,h_k,h_{k+2},\dots,h_{n+1},h_{k+1}). \end{aligned} \tag{5}$$

Hence A is a $\mathscr{C}^{n+1}$ mapping and

$$A^{(n+1)}(z)=\omega^{(n+1)}(z)\circ\tau_{n+3}(z)+\omega^{(n)}(z)\circ\tau_{n+2}(z)+\sum_{k=0}^{n-1}B_k'(z). \tag{6}$$

Let $C_k:\Omega\longrightarrow\mathscr{L}_{\mathbf{R}}(^{n+2}E;F)$, $k=1,\dots,n$, be defined by

$$C_k(z)(h_1,\dots,h_{n+2})=\omega^{(n)}(z)(h_1,\dots,h_k)\circ\tau_{n+2-k}(h_{k+1})(h_{k+2},\dots,h_{n+2}).$$

Define $C_0(z) = \omega^{(n)}(z) \circ \tau_{n+2}$. For $k = 1, \ldots, n$, we have the following equalities

$$\begin{aligned} &\omega^{(n)}(z)(h_1, \ldots, h_k) \circ \tau_{n+2-k}(h_{k+1})(h_{k+2}, \ldots, h_{n+2}) \\ &= [\omega^{(n)}(z)(h_1)][(h_2, \ldots, h_k) \circ \tau_{n+2-k}(h_{k+1})(h_{k+2}, \ldots, h_{n+2})] \\ &= \omega^{(n)}(z)(h_1)[(v_1, \ldots, v_{k-1}) \circ \tau_{n+1-(k-1)}(v_k)(v_{k+1}, \ldots, v_{n+1})] \end{aligned}$$

(where $v_{j-1} = h_j$ for all j, $2 \leqslant j \leqslant n+2$).

These equalities together with (5) imply that

$$C_k(z) = B'_{k-1}(z).$$

The last equality together with (6) imply that

$$A^{(n+1)}(z) = \omega^{(n+1)}(z) \circ \tau_{(n+1)+2}(z) + \sum_{k=0}^{n} C_k(z).$$

This proves (3) for $n+1$ and by induction this complete the proof. ∎

Now, we are in a position to prove our main result.

THEOREM 2.5. *Let E be a complex normed space and let Ω be an open subset of E. Let $\omega : \Omega \longrightarrow \Lambda^{(0,q)}(E)$ be a holomorphic $(0,q)$ form on Ω, $q \geqslant 1$. If $u : \Omega \longrightarrow \Lambda^{(0,q-1)}(E)$ is defined by $u(z) = \omega(z)(z)$, then u is a $\mathscr{C}^\infty$ $(0,q-1)$ form and $\overline{\partial} u = \omega$ on Ω.*

Proof. By the inclusion $\Lambda^{(0,q)}(E) \hookrightarrow \mathscr{L}({}^q\overline{E})$, we may suppose that $\omega : \Omega \longrightarrow \mathscr{L}({}^q\overline{E})$ is a $\mathscr{C}^\infty$ mapping. In fact $\omega : \Omega \longrightarrow \mathscr{L}_{\mathbf{R}}({}^qE; \mathscr{L}({}^{q-1}\overline{E}))$ and $u : \Omega \longrightarrow \mathscr{L}({}^{q-1}\overline{E})$. By (1), u is $\mathscr{C}^1$ and

$$u'(z) = \omega'(z) \circ \tau_2(z) + \omega(z). \tag{7}$$

Now, ω' can be seen as the following $\mathscr{C}^\infty$ mapping

$$\omega' : \Omega \longrightarrow \mathscr{L}_{\mathbf{R}}(E, \mathscr{L}_{\mathbf{R}}(E, \mathscr{L}_{\mathbf{R}}({}^{q-1}E))) \cong \mathscr{L}_{\mathbf{R}}({}^2E, \mathscr{L}_{\mathbf{R}}({}^{q-1}E)).$$

Consider the mapping

$$A : \Omega \longrightarrow \mathscr{L}_{\mathbf{R}}(E, \mathscr{L}_{\mathbf{R}}({}^{q-1}E)) \cong \mathscr{L}_{\mathbf{R}}({}^qE)$$

defined by $A(z) = \omega'(z) \circ \tau_2(z) = \omega'(z)(.,z) = \omega'(z)(.)(z)$.

By applying lemma 2.4, A is a $\mathscr{C}^\infty$ mapping. Hence, (7) shows that u is $\mathscr{C}^\infty$ on Ω. From (7) it follows easily that

(8) $\qquad [\bar\partial]u(z)(y) = [\bar\partial]\omega(z)(y,z) + \omega(z)(y) \qquad$ for $z \in \Omega$ and $y \in E$.

Since ω is holomorphic, $\omega'(z) \in \mathscr{L}_{\mathbb{C}}(E, \mathscr{L}(^q\bar{E}))$ and hence $[\bar\partial]\omega(z)(y,z) = 0$ for all $y \in E$. Hence, (8) can be written as

$$[\bar\partial]u(z) = \omega(z).$$

Since $\omega(z)$ is an alternating $q-$anti$-$linear form, we have $\bar\partial u = \omega$.

This complete the proof of theorem 2.5. ∎

We now solve the $\bar\partial-$problem for holomorphic $(0,q)$ forms, $q \geqslant 1$, on Fréchet–Montel and DFM spaces.

THEOREM 2.6. *Let E be a complex* DFM *or a complex Fréchet–Montel space and let $\omega : E \longrightarrow \Lambda^{(0,q)}(E)$, $q \geqslant 1$, be a holomorphic $(0,q)$ form on E. If $u : E \longrightarrow \Lambda^{(0,q-1)}(E)$ is defined by $u(z) = \omega(z)(z)$, then u is a $\mathscr{C}^\infty$ $(0,q-1)$ form on E and $\bar\partial u = \omega$ on E.*

Proof. a) Let E be a DFM space. By a result of Colombeau and Mujica [2] on factorization of holomorphic mappings from a DFM space into a metrizable locally convex space, we have for holomorphic $\omega : E \longrightarrow \Lambda^{(0,q)}(E)$, $q \geqslant 1$, that there exists a convex, balanced, open subset U of E such that ω factors as in diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \omega\ } & \Lambda^{(0,q)}(E) \\ {\scriptstyle \pi_U}\big\downarrow & & \big\downarrow{\scriptstyle ^q\pi_U} \\ E_U & \xrightarrow{\ \tilde\omega\ } & \Lambda^{(0,q)}(E_U) \end{array}$$

where E_U is the normed space associated with U, π_U is the canonical map, ${}^q\pi_U(\tilde A)(y_1, \dots, y_q) = \tilde A(\tilde y_1, \dots, \tilde y_q)$, $\pi_U(y_j) = \tilde y_j$ and $\tilde\omega$ is a holomorphic mapping of bounded type, i.e., bounded on the balls of E_U.

Theorem 2.5 implies that $\tilde u : E \longrightarrow \Lambda^{(0,q-1)}(E_U)$ defined by $\tilde u(\tilde z) = \tilde\omega(\tilde z)(\tilde z)$ is $\mathscr{C}^\infty$. In fact, $\tilde u$ is a $(0,q-1)$ form of bounded type and $\bar\partial\tilde u = \tilde\omega$ on E_U. Hence a $\mathscr{C}^\infty$ $(0,q-1)$ form $u : E \longrightarrow \Lambda^{(0,q-1)}(E)$ can be defined such that u is of uniform bounded type, i.e., the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\ u\ } & \Lambda^{(0,q-1)}(E) \\ {\scriptstyle \pi_U}\big\downarrow & & \big\downarrow{\scriptstyle ^{q-1}\pi_U} \\ E_U & \xrightarrow{\ \tilde u\ } & \Lambda^{(0,q-1)}(E_U) \end{array}$$

and $\bar{\partial}u = \omega$ on E. We refer to [7] (Section 5) for further details. It is easy to see that $u(z) = \omega(z)(z)$.

b) Now, we prove theorem 2.6 for a Fréchet–Montel space E. Following arguments given in [9] (lemma 3.3) we can show that a holomorphic $(0,q)$ form ω on E can be locally factorized through some E_U, i.e., for every $x \in E$ there exists a convex, balanced, open subset U of E such that ω factors as in the diagram

$$\begin{array}{ccc} x+U \subset E & \xrightarrow{\ \omega\ } & \Lambda^{(0,q)}(E) \\ \pi_U \downarrow & & \downarrow {}^{q}\pi_U \\ \tilde{x}+\tilde{U} \subset E_U & \xrightarrow{\ \tilde{\omega}_x\ } & \Lambda^{(0,q)}(E_U) \end{array}$$

where $\tilde{\omega}_x$ is a holomorphic $(0,q)$–form on $\tilde{x}+\tilde{U}$.

By theorem 2.5, $\tilde{u}_x : \tilde{x}+\tilde{U} \subset E_U \longrightarrow \Lambda^{(0,q-1)}(E_U)$ defined by $\tilde{u}_x(\tilde{z}) = \tilde{\omega}_x(\tilde{z})(\tilde{z})$ is $\mathscr{C}^\infty$ and $\bar{\partial}\tilde{u}_x = \tilde{\omega}_x$ in $\tilde{x}+\tilde{U}$.

Now $u : E \longrightarrow \Lambda^{(0,q-1)}(E)$ defined by $u(z) = \omega(z)(z)$ factors as in the following diagram

$$\begin{array}{ccc} x+U \subset E & \xrightarrow{\ u\ } & \Lambda^{(0,q-1)}(E) \\ \pi_U \downarrow & & \downarrow {}^{q-1}\pi_U \\ \tilde{x}+\tilde{U} \subset E_U & \xrightarrow{\ \tilde{u}_x\ } & \Lambda^{(0,q-1)}(E_U) \end{array}$$

In particular, u is $\mathscr{C}^\infty$, u factorizes locally through some E_U and $\bar{\partial}u = \omega$ on E. Theorem 2.6 extends to holomorphic $(0,q)$ forms, results given in [7], for holomorphic (0,2) forms on DFN spaces, and results in [9], for holomorphic (0,1) forms on Fréchet nuclear spaces.

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