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About the Interface of Some Nonlinear Diffusion Problems

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Let $P(m, C; u_0, u_1)$ denote the following problem:

(1) $u_t = (u^m)_{xx} + (C/(x+1)(u^m)_x \quad ((x,t) \in S = (0,\infty) \times (0,\infty),\ m > 0,\ C \geqslant 0)$

(2) $u(x,0) = u_0(x)$ for $x \in (0,\infty)$,

(3) $u(0,t) = u_1(t)$ for $t \in (0,\infty)$,

where

$$u_0 \in L^\infty(0,\infty),\quad \operatorname{ess\,inf} u_0 \geqslant 0,\quad u_0 \equiv 0 \text{ a.e. on } (\alpha,\infty) \quad (\alpha \geqslant 0), \tag{4}$$
$$u_1 \in L^\infty(0,\infty),\ \operatorname{ess\,inf} u_1 \geqslant \beta > 0.$$

Without loss of generality we can assume that $\beta = 1$.

In the case $C = 0$ equation (1) becomes the one–dimensional porous medium equation ([2], [3], [10]). If $C = N-1$ $(N = 2,3,\ldots)$ then (1) is the radial version of the N–dimensional porous medium equation $u_t = \Delta(u^m)$ transformed by introducing the translated spatial variable ([7]). Especially, the problem $P(2,1;0,1)$ describes the radially symmetrical infiltration into an unsaturated soil, when the level of water in a cylindrical reservoir is constant ([9]). The question of interest is the range of infiltrating water.

Under assumptions (4) the problem $P(m, C; u_0, u_1)$ has a unique weak solution $u = u(x,t)$ ([6], [7], [8]). The function u is nonnegative, bounded and continuous on S, and u satisfies an appropiate integral identity instead of (1). However u is the classical solution for those points $(x,t) \in S$ at which $u(x,t) > 0$. Moreover, if we define $\zeta(t) = \sup\{x \in (0,\infty) : u(x,t) > 0\}$ $(t > 0)$, then $0 < \zeta(t) < \infty$ for $t > 0$ and $\zeta(t)$ is a Lipschitz continuous nondecreasing function. The curve $x = \zeta(t)$ is called the interface or the free boundary of

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$P(m, C; u_0, u_1)$.

We know that in the case of $P(m,0;0,1)$ the interface has the form

$$\zeta(t) = c_0(m)t^{1/2} \quad , \tag{5}$$

where the constant $c_0(m) > 0$ depens on m ([1], [5], [11], [12], [16], [18], [19]).

If $C \in [0,1]$ then the interface of $P(m, C; u_0, u_1)$ satisfies the following asymptotic result ([17]):

$$\log \zeta(t) \sim \frac{1}{2}\log t \quad \text{as} \quad t \longrightarrow \infty . \tag{6}$$

In this paper we apply some integral equations methods ([4], [13], [14], [15]) to construct a so-called weak subsolution of $P(m, C; u_0, u_1)$ for $C > 1$ and use this subsolution to prove the following theorem

THEOREM. *Let $C > 1$. If ζ is the interface of the problem $P(m, C; u_0, u_1)$ then*

$$\zeta(t) \geqslant [(C-1)(C+1)m(m+1)^{-1}t + 1]^{1/(C+1)} - 1 \tag{7}$$

for $t \geqslant 0$.

In the authors' opinion the estimate (7) seems to be useful for further considerations concerning the large-time behavior of ξ.

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