EXTRACTA MATHEMATICAE Vol. 9, Núm. 1, 51 – 55 (1994)

# Existence Results for the Flow of Viscoelastic Fluids of White–Metzner Type

A. HAKIM

*Département de Mathématiques, Faculté des Sciences Guéliz, B.P.* 618 *Marrakech, Maroc*

AMS *Subject Class.* (1991): 33A05, 35K55, 35M05 *Received September* 1, 1993

## 1. INTRODUCTION

This work concerned with the study of the flow of an incompressible viscoelastic fluid of White–Metzner type (see [1, 2, 5]). These models lead to systems of partial differential equations that are evolutionary, are globally well posed (see [3]). The objective of this article is to prove the local and global existence of solutions of these systems. This paper is organized as follows. In the next section we formulate the mathematical problem. The third paragraph is devoted to local and global existence of solutions of the system for arbitrary data in some appropriate Sobolev space.

## 2. GOVERNING EQUATIONS

We examine parallel shear flow. The flow is incompressible. The total stress is composed of the pressure term $-p\underline{\underline{I}}$, plus the extra stress tensor $\underline{\underline{\tau}}$. For the White–Metzner model the extra stress is taken to be the sum of a polymeric contribution $\underline{\underline{\tau}}^p$, and an added Newtonian solvent term $\underline{\underline{\tau}}^s$, given by:

$$\begin{cases} \underline{\underline{\tau}}^p + \tilde{\lambda}_{II} \dfrac{\mathcal{D}\underline{\underline{\tau}}^p}{\mathcal{D}t} = 2\tilde{\eta}_{II}\, \underline{\underline{D}} & \text{(a)} \\ \underline{\underline{\tau}}^s = 2\eta_\infty\, \underline{\underline{D}} & \text{(b)} \end{cases} \tag{1}$$

where $\underline{\underline{D}}$ is the symmetric part of the velocity gradient tensor, $II$ is the second invariant of the rate of deformation tensor $(II := 2tr(\underline{\underline{D}}^2))$, $\tilde{\lambda}_{II}$ and $\tilde{\eta}_{II}$ are respectively the relaxation and viscosity functions which depend on $II$ (In the present work, we focus on a class of specific formula for $\tilde{\lambda}_{II}$ and $\tilde{\eta}_{II}$), $\eta_\infty > 0$ is the retardation time, the symbol $\frac{\mathcal{D}}{\mathcal{D}t}$ denotes an objective derivative (see [3]). This model includes many of the classical constitutive equations as special cases (see [2, 3]). We assume that the flow variables are independent of $y$. Therefore the velocity field is $\underline{U} = (0, v(x,t))$ where $x \in I = \left[\frac{-1}{2}, \frac{1}{2}\right]$ and the balance of mass is automatically satisfied. Futhermore, the components of each partial stress tensor $\underline{\underline{\tau}}^p$ can be writen

$$\tau^p_{xx} := \gamma(x,t) \qquad \tau^p_{xy} = \tau^p_{yx} := \sigma(x,t) \qquad \tau^p_{yy} := \tau(x,t) .$$

The pressure takes the form $p = -p_0(x,t) - f(t)y$, $f$ being the pressure gradient driving the flow. In these terms, the dimensionless time–dependent parallel shear flow equations are given by (for the detail see [2, 3])

$$\begin{cases} \hat{R}e\, v_t - \sigma_x = \theta v_{xx} + f & \text{(a)} \\ \sigma_t - [Z + \mu_{II}]\, v_x = -\sigma / \lambda_{II} & \text{(b)} \\ Z_t + \sigma v_x = -Z / \lambda_{II} & \text{(c)} \end{cases} \qquad (2)$$

where $\omega = 1 - \tilde{\eta}$ is the viscosity ratio, $\tilde{\eta} = \eta_\infty / \eta_0$, $\mu_{II} = \eta_{II} / \lambda_{II}$, $\hat{R}e = Re/\omega$ is the Renolds number, $\theta = (1 - \omega)/\omega$, $Z := (1 - \epsilon)\gamma - \epsilon\tau$ $(0 \leqslant \epsilon \leqslant 1)$ and

$$\lambda_{II} = \frac{We}{[1 + We^2 II]^\beta} , \qquad \eta_{II} = \frac{1 + (\lambda_{II})^2 II}{[1 + We^2 II]^\alpha} . \qquad (3)$$

The boundary and initial conditions are given by (see [6])

$$\begin{aligned} & v(\tfrac{-1}{2}, t) = 0 , \qquad v(\tfrac{1}{2}, t) = 0 && \text{(a)} \\ & v(x,0) = v_0(x) , \quad \sigma(x,0) = \sigma_0(x) , \quad Z(x,0) = Z_0(x) && \text{(b)} . \end{aligned} \qquad (4)$$

## 3. Local and Global Existence of Regular Solutions

In this section we implement a fixed point argument to show the existence of a regular solution to problem (2) – (4) on a small interval $[0, T^*]$.

THEOREM 1. *Assume* $f \in L^2_{loc}(\mathbb{R}_+, H^1)$, $f' \in L^2_{loc}(\mathbb{R}_+, H^{-1})$, $Z_0 \in H^2$, $\sigma_0 \in H^2(I)$ *and* $v_0 \in H^2(I) \cap H^1_0(I)$ . *Then there exists a* $T^* > 0$,

$$v \in L^2(0, T^*, H^3) \cap C\big([0, T^*], H^2 \cap H^1_0\big) , \quad v' \in L^2(0, T^*, H^1) \cap C\big([0, T^*], L^2\big)$$

$$(\sigma, Z) \in \Big[C\big([0, T^*], H^2\big)\Big]^2 , \qquad (\sigma', Z') \in \Big[L^\infty\big([0, T^*], H^1\big)\Big]^2$$

*such that* $(v,\sigma,Z)$ *is a solution to problem* (2) – (4).

*Idea of the proof.* The proof of Theorem 1 is obtained by Schauder's fixed point theorem. We first study two linearized problems, one for the velocity $v$ and the other for the polymeric stress (see [2, 3, 7]).

(i) Let $T$ be a given positive real number, we consider the following linear problem: To find $v$ such that

$$\begin{cases} \hat{Re}\, v_t - \theta v_{xx} = F \\ v(x,0) = v_0(x) \end{cases} \tag{5}$$

where $F$ is a given external body force.

The proof of the existence of the solution of this system is classical. For similar calculation see [2, 3, 7].

(ii) We turn now to the study of linearized problem associated to the constitutive equation.

Let $\tilde{v} \in L^2(0,T^*,H^3) \cap C([0,T^*],H^2 \cap H_0^1)$ is given. We consider the following system

$$\begin{cases} \sigma_t + (\sigma/\lambda_{II}) = [Z + \mu_{II}]\,\tilde{v}_x \\ Z_t + (\sigma/\lambda_{II}) = -\,\sigma\tilde{v}_x \\ \sigma(x,0) = \sigma_0(x)\ ,\quad Z(x,0) = Z_0(x) \end{cases} \tag{6}$$

and we show that $\sigma$ and $Z$ satisfies the following estimates

$$\left\{ \|\sigma\|^2_{L^\infty(0,T,H^2)} + \|Z\|^2_{L^\infty(0,T,H^2)} \right\} \leqslant$$

$$C\left(1 + \|\sigma_0\|_2^2 + \|Z_0\|_2^2\right) \times \left(1 + \|\tilde{v}\|^2_{L^2(0,T,H^3)}\right)^2 \times \left(1 + \|\tilde{v}\|^2_{L^\infty(0,T,H^2)}\right)^2$$

$$\|\sigma'\|_1^2 \leqslant C\left(1 + \|\tilde{v}\|^2_{L^\infty(0,T,H^2)}\right)^2 \left(1 + \|\sigma\|_1^2 + \|Z\|_1^2\right)$$

$$\|Z'\|_1^2 \leqslant C\left(1 + \|\tilde{v}\|^2_{L^\infty(0,T,H^2)}\right)^2 \left(1 + \|\sigma\|_1^2 + \|Z\|_1^2\right)$$

where $C = C(I,\lambda,\mu)$. For the detail see [3, 7].

(iii) We consider the mapping

$$\Phi :\ R_T \longrightarrow X_T := C([0,T],H^2 \cap H_0^1) \times \left[C([0,T],H^1)\right]^2$$

$$(\tilde{v},\tilde{\sigma},\tilde{Z}) \longmapsto (v,\sigma,Z)$$

where $v$ and $(\sigma,Z)$ are the unique solution of (5) and (6) respectively, with $F = \tilde{\sigma}_x + f$ and $R_T$ is define by

$$
\begin{aligned}
R_T = \Big\{ \quad & (\tilde{v},\tilde{\sigma},\tilde{Z}) : \tilde{v} \in L^2(0,T,H^3) \cap C\big([0,T], H^2 \cap H_0^1\big) , \\
& \tilde{v}' \in L^\infty(0,T,L^2) \cap L^2(0,T,H^1) , \\
& (\tilde{\sigma},\tilde{Z}) \in \left[L^\infty(0,T,H^2)\right]^2 , \quad (\tilde{\sigma}',\tilde{Z}') \in \left[L^\infty(0,T,H^1)\right]^2 , \\
& \|\tilde{v}\|^2_{L^\infty(0,T,H^2)} + \|\tilde{v}\|^2_{L^2(0,T,H^3)} + \|\tilde{v}'\|^2_{L^\infty(0,T,L^2)} + \|\tilde{v}'\|^2_{L^2(0,T,H^1)} \leqslant B_1 , \\
& \|\tilde{\sigma}\|^2_{L^\infty(0,T,H^2)} + \|\tilde{Z}\|^2_{L^\infty(0,T,H^2)} \leqslant B_2 , \\
& \|\tilde{\sigma}'\|^2_{L^\infty(0,T,H^1)} + \|\tilde{Z}'\|^2_{L^\infty(0,T,H^1)} \leqslant B_2 , \\
& \tilde{v}(0) = v_0 , \ \tilde{\sigma}(0) = \sigma_0 , \ \tilde{Z}_0 = Z_0 \text{ in } I \quad \Big\} .
\end{aligned}
$$

And we show that if $B_1$ and $B_2$ are large enough, then $R_T \neq \emptyset$ for all $T > 0$ see [2, 3].

A fixed point of $\Phi$ is clearly a solution of (2) – (4). First we prove the existence of $T^*$ small enough so that $\Phi(R_{T*}) \subset (R_{T*})$. The conclusions of Theorem 1 follow from Schauder's fixed point theorem applied to the mapping $\Phi$ on the convex set $R_{T*}$. Indeed it is easly to show that $R_{T*}$ is closed in $X_{T*}$. Moreover, by Ascoli's theorem, $R_{T*}$ is compact in $X_{T*}$. Standard arguments prove that $\Phi$ is continuous for the topology of $X_{T*}$ (for the details see [3, 7]).

We now state the theorem of existence of a global solution.

THEOREM 2. *There exist some $\theta_0$, $0 < \theta_0 < 1$, depending on $I$, $\lambda$ and $\mu$ such that if $\theta_0 < \theta \leqslant 1$ and if $v_0 \in H^2 \cap H_0^1$, $\sigma_0 \in H^1$, $Z_0 \in H^1$, $f \in L^\infty(\mathbb{R}_+,H^1)$ and $f' \in L^\infty(\mathbb{R}_+,H^{-1})$ are small enough in their spaces, then the problem* (2) – (4) *admits a unique solution $(v,\sigma,Z)$ defined for all times $t$ and satisfies*

$$
\begin{gathered}
v \in C_b(\mathbb{R}_+,H^2 \cap H_0^1) \cap L^2_{loc}(\mathbb{R}_+,H^3) , \quad v' \in C_b(\mathbb{R}_+,L^2) \cap L^2_{loc}(\mathbb{R}_+,H^1) , \\
(\sigma,Z) \in \left[C_b(\mathbb{R}_+,H^2)\right]^2 , \quad (\sigma',Z') \in \left[C_b(\mathbb{R}_+,H^1)\right]^2 .
\end{gathered}
$$

*Idea of the proof.* We show that the local solution obtained in Theorem 1 is actually define on $\mathbb{R}_+$ if the data are small enough. To this end we derive some a priori bounds uniform in time, satisfied by the solution. For the details see [2, 3, 7].

*Remarks.* (i) Condition $\theta_0 < \theta \leqslant 1$ means that the fluid has a certain amount of Newtonian viscosity.

(ii) No result such as Theorem 2 seems to be known for $\theta = 0$. We have however to mention Kim [4] where the upper–convected Maxwell model in the whole space $\mathbb{R}_+$ is considered.

## REFERENCES


1. GAIDOS, R.E. AND DARBY, R., Numerical simulation and change in type in the developing flow of a nonlinear viscoelastic fluid, *Journal of Non–Newtonian Fluid Mech.* **29** (1988), 59 – 79.
2. HAKIM, A., Mathematical analysis of viscoelastic fluids of White–Metzner type, *Jour. of Math. Anal. and Appl.*, to appear.
3. HAKIM, A., Thèse, Université Paris–Sud Centre d'Orsay (1989).
4. KIM, J.U., Global smooth solutions for the equations of motion of a nonlinear fluid with fading memory, *Arch. Rat. Mech. Anal.* **79** (1982), 97 – 130.
5. LARSON, R.G., "Constitutive Equations for Polymer Melts and Solutions", Butterworts Series in Chemical Engineering (1987).
6. MALKUS, D.S., NOHEL, J.A. AND PLOHR, B.J., Dynamics of shear flow of a non–newtonian fluid, *J. Comput. Phys.* **87** (1990), 464 – 487.
7. VALLI, A., Periodic and stationary solutions for compressible Navier–Stokes equations via a stability method, *Annali Scu. Norm. Sup. Pisa* **10** (1983), 607 – 647.