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Classification of Symmetries for Higher Order Lagrangian Systems [1]

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The purpose of this paper is to give a complete classification of the infinitesimal symmetries of a higher order Lagrangian system (see [9]). Our classification extends the one obtained by Prince [16,17] for first-order Lagrangian mechanics (see also Crampin [4], de León and Rodrigues [12], Crampin, Sarlet and Cantrijn [5], Cariñena, López and Martínez [2]). Symmetries of higher order Lagrangians systems were also studied by Grigore [6,7,8] but using a different geometric formulation of Lagrangian mechanics, based in that of Souriau [20]. We shall use the symplectic formulation of higher order of Lagrangian mechanics [10,11,5] and the theory of lifts of functions and vector fields to higher order tangent bundles [14,23].

A Lagrangian of order k is a function $L=L(q_0^a,q_1^a,\dots,q_k^a)$ which depends on the position variables q_0^a and its derivatives up to order k (see [22] for a classical reference, and [3,18,19,1,13] for some examples). Using higher order tangent bundles, we may consider L as a function $L: T^kQ \longrightarrow \mathbb{R}$. We say that L is regular if the Hessian matrix $(\partial^2 L/\partial q_k^a \partial q_k^b)$ is of maximal rank. We denote by E_L the energy associated to a regular Lagrangian L of order k. Let α_L be the Poincaré–Cartan 1-form and $\omega_L = -d\alpha_L$ the Poincaré–Cartan 2-form. The intrinsic expressions of E_L and α_L are the following:

$$\begin{aligned} E_L &= \sum_{r=1}^{k} (-1)^{r-1} \frac{1}{r!} \left[\tau_{k+r-1}^{2k-1}\right]^* (d_T^{r-1}(C_r L)) - \left[\tau_k^{2k-1}\right]^* L, \\ \alpha_L &= \sum_{r=1}^{k} (-1)^{r-1} \frac{1}{r!} \left[\tau_{k+r-1}^{2k-1}\right]^* d_T^{r-1}(d_{J_r}) L, \end{aligned}$$

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where d_T is the total derivative with respect to the time (see Tulczyjew [21], de León and Rodrigues [11]), $\tau_s^r : T^rQ \longrightarrow T^sQ$ is the canonical projection, J_1 is the canonical higher order almost tangent structure, $J_r = (J_1)^r$, C_1 is the higher order Liouville vector field and $C_r = J_{r-1}C_1$. The global equation of the motion may be written on the generalized velocity phase space $T^{2k-1}Q$ as

$$i_X\omega_L = dE_L \tag{1}$$

In fact, since ω_L is symplectic, then there exists a unique vector field ξ_L on $T^{2k-1}Q$, which satisfies (1). ξ_L will be called the Euler– Lagrange vector field. Furthermore, ξ_L is a $2k$th–order differential equation and its solution are just the solutions of the Euler–Lagrange equations for L (see [10,11]):

$$\sum_{r=0}^{k}(-1)^r\frac{d^r}{dt^r}\left[\frac{\partial L}{\partial q_r^i}\right] = 0\,. \tag{2}$$

The existence of constants of the motion is useful in order to integrate the motion equations (2) (see Olver [15]). Let us recall that a differentiable function $f : T^{2k-1}Q \longrightarrow \mathbb{R}$ is said to be a constant of the motion if $\xi_L f = 0$. In other words, if $\gamma : I \longrightarrow T^{2k-1}Q$ is an integral curve of ξ_L then $f\circ\gamma$ is a constant function.

If $\varphi : Q \longrightarrow Q$ is a mapping, we denote by $T^r\varphi$ its natural lift to the tangent bundle of order r, i.e., $T^r\varphi : T^rQ \longrightarrow T^rQ$. Also, if X is a vector field on Q, we denote by $X^{(r,r)}$ its natural lift to T^rQ (see [14,23]).

We may distinguish the following three types of symmetries of the Lagrangian system defined by L:

1. A diffeomorphism $\phi : T^{2k-1}Q \longrightarrow T^{2k-1}Q$ is said to be a symmetry of ξ_L if $T\phi(\xi_L) = \xi_L$.
2. A diffeomorphism $\varphi : Q \longrightarrow Q$ is said to be a point symmetry of ξ_L if $T^{2k-1}\varphi$ is a symmetry of ξ_L.
3. A diffeomorphism $\varphi : Q \longrightarrow Q$ is said to be a symmetry of L if $L \circ T^k\varphi = L$.

We obtain a classification of infinitesimal symmetries in two classes, namely point–symmetries (vector fields on Q) and infinitesimal symmetries not necessarily point–like (vector fields on $T^{2k-1}Q$). The point–symmetries are classified as follows:

Let X be a vector field on Q.

1. X is said to be a Lie symmetry if $[\xi_L, X^{(2k-1,2k-1)}] = 0$, or, equivalently,

its flow consists of point symmetries of ξ_L.

2. X is said to be a Noether symmetry if $L_{X^{(2k-1,2k-1)}}\alpha_L$ is exact (i.e., $L_{X^{(2k-1,2k-1)}}\alpha_L = df$) and $X^{(2k-1,2k-1)}E_L = 0$.
3. X is said to be an infinitesimal symmetry of L if $X^{(k,k)}L = 0$, or, equivalently, its flow consists of symmetries of L.

Next, we give the classification of the symmetries not necessarily point–like. Let $\tilde{X}$ be a vector field on $T^{2k-1}Q$.

1. $\tilde{X}$ is said to be a dynamical symmetry if $[\xi_L, \tilde{X}] = 0$, or equivalently, its flow consists of symmetries of ξ_L.
2. $\tilde{X}$ is said to be a Cartan symmetry if $L_{\tilde{X}}\alpha_L$ is exact, i.e., $L_{\tilde{X}}\alpha_L = df$, and $\tilde{X}E_L = 0$.

We have obtained the following results which relate the different types of infinitesimal symmetries:

1. A Noether symmetry is a Lie symmetry.
2. An infinitesimal symmetry of L is a Noether symmetry.
3. A Cartan symmetry is a dynamical symmetry.
4. If X is a Noether symmetry, then $X^{(2k-1,2k-1)}$ is a Cartan symmetry.
5. If X is a Lie symmetry, then $X^{(2k-1,2k-1)}$ is a dynamical symmetry.

The relationship between infinitesimal symmetries and constants of the motion is given in the following results:

1. Let X be an infinitesimal symmetry of L. Then

$$\alpha_L(X^{(2k-1,2k-1)}) = \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}\left[T^{2k-1}_{k+r-1}\right]^*(d_T^{r-1}(X^{(k-r,k)}L))$$

is a constant of the motion.

2. If X is a Noether symmetry, then

$$f - \alpha_L(X^{(2k-1,2k-1)}) = f - \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}\left[T^{2k-1}_{k+r-1}\right]^*(d_T^{r-1}(X^{(k-r,k)}L))$$

is a constant of the motion.

3. (Noether theorem and its converse) If $\tilde{X}$ is a Cartan symmetry of ξ_L then $f - \alpha_L(\tilde{X})$ is a constant of the motion. Conversely, if f is a constant of the motion and Z is a Hamiltonian vector field on $T^{2k-1}Q$

such that $i_Z \omega_L = df$, then Z is a Cartan symmetry.

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