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On the Mehler–Fock Integral Transform in L_p-Spaces

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This paper deals with the special kind of the known Mehler–Fock integral transform [1], which first is investigated in the weighted $L_{\nu,p}(\mathbb{R}_+)$ spaces with

$$\int_0^\infty |t^\nu f(t)|^p \frac{dt}{t} < \infty , \quad (1 \leqslant p < \infty ;\ \nu \in \mathbb{R}) .$$

The L_p-theory of the Mehler–Fock transform based on its composition structure and the related results on the Kontorovich–Lebedev and the Hankel transforms in L_p (the details have been described in [4], see also [3]).

Let us consider the following integral representation

$$\frac{\pi}{2\cosh(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) = \int_0^\infty J_0(xy)\, K_{i\tau}(y)\, dy ,$$

where $P_\nu(z)$ is the Legendre function, $J_\nu(z)$ is the Bessel function and $K_\nu(z)$ is the MacDonald function. As it is well known, these functions are the kernels of the following Mehler–Fock, Hankel and Kontorovich–Lebedev integral transforms respectively

$$[P_{-1/2+i\tau/2} f] = \frac{\pi\tau}{2\cosh(\pi\tau/2)} \int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)\sqrt{y} f(y)\, dy ,$$

$$[J_0 f](x) = \int_0^\infty \sqrt{xy}\, J_0(xy) f(y)\, dy ,$$

$$[K_{i\tau} f] = \tau \int_0^\infty K_{i\tau}(y) f(y)\, dy .$$

In order to obtain the inversion theorem of the Mehler–Fock transform in L_p we need the suitable estimate of the Legendre function. Using the integral representation of the MacDonald function

$$K_{i\tau}(x) = \frac{1}{2} \int_{\delta i - \infty}^{\delta i + \infty} e^{-x\cosh(\beta)} e^{i\tau\beta} \, d\beta \,, \quad x > 0 \,, \quad \delta \in [0, \pi/2) \,,$$

substituting it in the formula above for the Legendre function and interchanging the order of integration due to the absolute convergence of the iterated integral we shall have

$$\frac{\pi}{\cosh(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) = \int_{\delta i - \infty}^{\delta i + \infty} \frac{e^{i\tau\beta}}{\sqrt{x^2 + \cosh^2(\beta)}} \, d\beta \,.$$

Hence, we shall have the following useful estimate

$$\frac{1}{\cosh(\pi\tau/2)} |P_{-1/2+i\tau/2}(2x^2+1)| \leqslant e^{-\delta\tau} P_{-1/2}(2x^2+1) \,, \; \tau > 0 \,, \; x > 0 \,, \; \delta \in [0, \pi/2) \,.$$

The main result of this announcement is contained in the next

THEOREM. *If* $4/3 \leqslant p < \infty$, *then the operator of the Mehler–Fock transform is bounded operator from* $L_{3/2-1/p,p}(\mathbb{R}_+)$ *to* $L_r(\mathbb{R}_+)$, *where* $1 \leqslant r < \infty$, *it is one–to–one transform and the following composition is true*

$$[P_{-1/2+i\tau/2} f] = [K_{i\tau} \frac{1}{\sqrt{x}} [J_0 f](x)] \,.$$

Moreover, for $4/3 \leqslant p \leqslant 2$ *we have also the next inversion formula*

$$f(x) = \sqrt{x} \, \lim_{\epsilon \to 0+} \frac{1}{\pi} \int_0^\infty \frac{\sinh((\pi - \epsilon)\tau)}{\cosh(\pi\tau/2)} P_{-1/2+i\tau/2}(2x^2+1) \, [P_{-1/2+i\tau/2} f] \, d\tau \,,$$

where the limit is understanding by the norm of $L_{3/2-1/p,p}(\mathbb{R}_+)$.

Proof. Making use the estimate and asymptotic behaviour of the Legendre function with the general Minkowski inequality we obtain

$$\|[P_{-1/2+i\tau/2} f]\|_{L_r} \leqslant \int_0^\infty P_{-1/2}(2x^2+1) \sqrt{x} f(x) \left\{ \int_0^\infty \tau^r e^{-\delta r\tau} \, d\tau \right\}^{1/r} dx \leqslant$$

$$\leqslant C \left[\int_0^1 \sqrt{x} f(x) \, dx + \int_1^\infty \frac{1}{\sqrt{x}} f(x) \, dx \right] <$$

$$< C \left[\int_0^1 x^{(2/p-1)q} \, dx \right]^{1/q} \|f\|_{L_{3/2-1/p,p}} +$$

$$+ \; C\left[\int_1^\infty x^{(1/p-1)2q}\,\mathrm{d}x\right]^{1/q}\|f\|_{L_{3/2-1/p,p}} < C_1\,\|f\|_{L_{3/2-1/p,p}}\,,\; 1/p+1/q=1\,,$$

where C, C_1 are absolute positive constants and two last inequalities have been obtained with the Hölder inequality. Moreover, we proved the validity of announced composition for the Mehler–Fock transform as the Kontorovich–Lebedev transform from the Hankel transform of zero index, because the respective iterated integral is absolute convergent one and we can interchange the order of integration. Since due to [3] the Hankel transform is bounded operator and one–to–one transform from the spaces $L_{3/2-1/p,p}(\mathbb{R}_+) \longrightarrow L_{1/p-1/2,p}(\mathbb{R}_+)$ or $1/\sqrt{x}\,[J_0f](x) \in L_p(\mathbb{R}_+)$, then due to [4] the Mehler transform is one–to–one bounded operator from $L_{3/2-1/p,p}(\mathbb{R}_+)$ to $L_r(\mathbb{R}_+)$, $1 \leqslant r < \infty$. After inversion of the Kontorovich–Lebedev transform in $L_p(\mathbb{R}_+)$ we shall have

$$[J_0f](x) = \lim_{\epsilon\to 0+}\frac{2}{\pi}\int_0^\infty \sinh((\pi-\epsilon)\tau)\,(K_{i\tau}(x)/\sqrt{x})\,[P_{-1/2+i\tau/2}f]\,\mathrm{d}\tau\,,$$

where the limit is understanding by the $L_{1/p-1/2,p}$–norm. But if $4/3 \leqslant p \leqslant 2$, then we can invert the Hankel transform (see [3]) and applying the above integral representation for the Legendre function to carry out the sign of the limit from the Hankel operator and interchange the order of integration due to above estimates for obtaining the announced inversion formula for the Mehler–Fock transform in L_p–spaces.

REFERENCES

1. FOCK, V.A., On the representation of the ordinary function in the integral by the Legendre function with a complex index, *Dokl. Akad. Nauk. SSSR* **39**(7) (1943), 279–283 (in Russian).
2. PRUDNIKOV, A.P., BRYCHKOV, YU.A. AND MARICHEV, O.I., "Integrals and Series", Gordon and Breach, New York, Vol. 1, 2, 1986, Vol. 3, 1989.
3. ROONEY, P.G., On integral transformations with G–function kernels, *Proc. Royal Soc. Edinburgh* Sec. A **93** (1982/83), 265–297.
4. YAKUBOVICH, S.B. AND LUCHKO, YU.F., "Hypergeometric Approach to Integral Transforms and Convolutions", Kluwer Academic Publishers, Ser. Mathematics and Its Applications, Dordrecht, (to appear).