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On the λ–Property and Computation of the λ–Function of some Normed Spaces

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0. INTRODUCTION

R.M. Aron and R.H. Lohman introduced, in [1], the notion of λ–property in a normed space and calculated the λ–function for some classical normed spaces. In this paper we give some more general remarks on this λ–property and compute the λ–function of other normed spaces namely: $B(S,\Sigma,X)$ and $M_d(E)$.

If X is a normed space, the closed unit ball and the unit sphere will be denoted by B_X and S_X respectively. The set of extreme points of B_X is denoted by $\mathrm{ext}(B_X)$. Recall that X is strictly convex if $\mathrm{ext}(B_X)=S_X$. Let $x\in B_X$, if $e\in \mathrm{ext}(B_X)$, $\|y\|\leqslant 1$, $0<\lambda\leqslant 1$ and $x=\lambda e+(1-\lambda)y$ we say (cf. [1]) the ordered triple (e,y,λ) is *amenable to* x. In this case, we define (cf. [1])

$$\lambda(x)=\sup\{\lambda : (e,y,\lambda) \text{ is amenable to } x\}\,.$$

Recall that X is said to have the *λ–property* if each $x\in B_X$ admits an amenable triple. If X has the λ–property and, in addition, satisfies $\inf\{\lambda(x) : x\in B_X\}>0$, we say X has the *uniform λ–property*.

If T is a compact Hausdorff space, we denote by $C_X(T)$ the normed space of all continuous functions on T valued in X with the norm sup.

1. SOME RESULTS ON THE λ-FUNCTION IN A NORMED SPACE

PROPOSITION 1.1. *Let X be a normed space having the λ–property (resp. uniform λ–property). Let Y be a normed space and let $f\colon X\longrightarrow Y$ an isometric isomorphism. Then Y has the λ–property (resp. uniform λ–property) and:*

$$\lambda_X(x)=\lambda_Y\circ f(x)\,,\quad \forall\ x\in B_X\,;$$

where λ_X (resp. λ_Y) is the λ–function of X (resp. Y).

Proof. Easy. ∎

LEMMA 1.2. *Let X be a normed space having the λ–property and let $x \in B_X$. If $\lambda(x) = 1$, then $x \in \overline{\mathrm{ext}(B_X)}$.*

Proof. If $\lambda(x) = 1$, then for each $n \in \mathbb{N}^*$, there exists a triple (e_n, y_n, λ_n) amenable to x such that $1 - 1/n < \lambda_n \leqslant 1$. Then $e_n = x/\lambda_n + (1 - 1/\lambda_n)y_n$. Hence the sequence (e_n) converges to x. ∎

For the λ–function of the space $C_X(T)$ we have:

LEMMA 1.3. *Let T be a compact Hausdorff space and let X be a normed space. If $C_X(T)$ has the λ–property, then we have:*

$$^1/_2(1 + m)\,\lambda(z) \leqslant \lambda(x) \leqslant {}^1/_2(1 + m)\,,$$

where $m = \inf\{\|x(t)\| : t \in T\}$ and $z(t) = x(t)/\|x(t)\|$ for all x in the closed unit ball of $C_X(T)$ such that $x(t) \neq 0$ for all $t \in T$.

Proof. It is easy to see (cf. [1]) that $\lambda(x) \leqslant (1 + m)/2$. The case $m = 1$ is trivial, so we assume $m < 1$. Write $z(t) = x(t)/\|x(t)\|$ and

$$y(t) = \frac{2\|x(t)\| - 1 - m}{(1 - m)\|x(t)\|}\, x(t)$$

for all $t \in T$. We have $\|y\| \leqslant 1$ and $x = {}^1/_2(1 + m)z + {}^1/_2(1 - m)y$.

Given $\epsilon > 0$, there is a triple (e, y', λ) that is amenable to z for wich $\lambda(z) - \epsilon < \lambda$. Letting $\lambda' = {}^1/_2(1 + m)\lambda$ and

$$y'' = \frac{(1 + m)(1 - \lambda)y' + (1 - m)y}{2 - (1 + m)\lambda}\,.$$

Then (e, y'', λ') is amenable to x. This shows $\lambda(x) > {}^1/_2(1 + m)(\lambda(z) - \epsilon)$. Completing the proof. ∎

Remark 1.4. Consequently, if X is a normed space having the λ–property, we have (see also [1]):

$$^1/_2(1 + \|x\|)\,\lambda\big(x/\|x\|\big) \leqslant \lambda(x) \leqslant {}^1/_2(1 + \|x\|) \qquad \text{for all } x \in B_X \backslash \{0\}.$$

Since if T is a single set, there is an isometric isomorphism from $C_X(T)$ onto X.

THEOREM 1.5. *Let X be a normed space and T be a compact Hausdorff space. Denote $Y = C_X(T)$ and $E = \{x \in Y : \|x(t)\| = 1 \text{ for all } t \in T\}$. Assume that*

Y has the λ-property. If $\lambda(x)=1$ for all $x \in E$ and $\operatorname{ext}(B_Y)$ *is closed, then* $\operatorname{ext}(B_Y)=E$ *and* $\lambda(x)={}^1/_2(1+m)$ *for all $x \in B_Y$; where as usual* $m=\inf\{\|x(t)\| : t \in T\}$.

Proof. By the Lemma 1.4 of [1] we have $\operatorname{ext}(B_Y) \subset E$ and by our Lemma 1.2 we have $E \subset \overline{\operatorname{ext}(B_Y)}$. Using Lemma 1.3 we get $\lambda(x)={}^1/_2(1+m)$ for all $x \in B_Y$. ∎

COROLLARY. *Let X be a normed space having the λ-property. Then X is strictly convex if and only if $\lambda(x)=1$ for all $x \in S_X$ and* $\operatorname{ext}(B_X)$ *is closed.*

2. COMPUTATION OF THE λ-FUNCTION OF THE SPACE $B(S,\Sigma,X)$

2.1. Let S be an arbitrary set and let Σ be an algebra of subsets of S. Let X be a normed space. For $x \in X$ and $A \in \Sigma$, we call the function $s \longrightarrow \chi_A(s)x$ a *X-characteristic function*, where χ_A is the usual characteristic function of A. The space $B(S,\Sigma,X)$ consists of all uniform limits of finite sums of X-characteristic functions. The norm in $B(S,\Sigma,X)$ is given by:

$$\|f\| = \sup\{\|f(s)\| : s \in S\}.$$

We note that, if $f \in B(S,\Sigma,X)$ with $\inf\{\|f(s)\| : s \in S\} \neq 0$, then the function $s \longmapsto f(s)/\|f(s)\|$ is also in $B(S,\Sigma,X)$.

LEMMA 2.2. *Let S be an arbitrary set and let X be a normed space. If e is an extreme point of the closed unit ball of $B(S,\Sigma,X)$, then $\|e(s)\|=1$ for all $s \in S$.*

Proof. Suppose there exists $s_0 \in S$ such that $\|e(s_0)\|=\alpha<1$. Let $\delta=(1-\alpha)/4$ and set $V=\{s \in S : \|e(s)\| \leqslant \alpha+\delta\}$. Then $s_0 \in V$. Fix $x_0 \in S_X$ and define $u, v \in B(S,\Sigma,X)$ by:

$$u(s)=e(s)+\delta\chi_V(s)x_0\,, \quad v(s)=e(s)-\delta\chi_V(s)x_0\,.$$

Then $e={}^1/_2(u+v)$, contradicting the fact that e is an extreme point of the closed unit ball of $B(S,\Sigma,X)$. ∎

Remark 2.3. If $e \in B(S,\Sigma,X)$ and $e(s) \in \operatorname{ext}(B_X)$ for all $s \in S$, then e is an extreme point of the closed unit ball of $B(S,\Sigma,X)$. Consequently, if X is a strictly convex normed space, then the converse of Lemma 2.2 is true.

THEOREM 2.4. *Let S be an arbitrary set and let X be a strictly convex normed space. Then $B(S,\Sigma,X)$ has the uniform λ-property. In fact, if*

$x \in B(S,\Sigma,X)$ and $\|x\| \leqslant 1$, then $\lambda(x) = {}^1/_2(1+m)$, where $m = \inf\{\|x(s)\| : s \in S\}$. Moreover, if $m \neq 0$, then $\lambda(x)$ is attained.

Proof. One proceeds exactly as in the proof of Theorem 1.6 of [1], noting that only the case in which $m = 0$ needs to be modified. In this case, let $0 < \lambda < {}^1/_2$, choose $\delta > 0$ such that $4\delta < 1 - 2\lambda$ and let $W = \{s \in S : \|x(s)\| \geqslant 2\delta\}$. Fix $x_0 \in X$, $\|x_0\| = 1$, and define $e : S \longrightarrow S_X$ by:

$$e(s) = \big(f(s)/\|f(s)\|\big)\chi_W(s) + \chi_{W^c}(s)x_0 ,$$

where $f(s) = x(s)\chi_W(s) + \chi_{W^c}(s)x_0$ and $W^c = S \setminus W$.

Then e is an extreme point of the closed unit ball of $B(S,\Sigma,X)$. Define $y \in B(S,\Sigma,X)$ by $y = (x - \lambda e)/(1-\lambda)$. Since (e,y,λ) is amenable to x, and $0 < \lambda < {}^1/_2$ is arbitrary, we have $\lambda(x) \geqslant {}^1/_2$. ∎

Remark. If Σ is the algebra of all subsets of S, then $B(S,\Sigma,X)$ is the space of all bounded functions defined in S, valued in X.

3. Computation of the λ-Function of the Space $M_d(E)$

Let E be an (infinite) locally compact space. A complex measure $\mu \in M(E)$ is called *purely discontinuous*, if there exists a countable subset F of E such that $|\mu|(F^c) = 0$. The space of all such measure will be denoted by $M_d(E)$.

For $a \in E$, let ϵ_a be the measure defined by $\epsilon_a(A) = \chi_A(a)$ for all $A \subset E$. We recall the following result (see [6], p. 270).

THEOREM 3.1. *For $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of complex numbers such that $\Sigma_{n=1}^{\infty}|\alpha_n| < \infty$ and $\{a_n\}_{n=1}^{\infty}$ a sequence of distinct points of E, we have:*

(i) $\Sigma_{n=1}^{\infty}\alpha_n\epsilon_{a_n} \in M_d(E)$;

(ii) $|\Sigma_{n=1}^{\infty}\alpha_n\epsilon_{a_n}| = \Sigma_{n=1}^{\infty}|\alpha_n|\epsilon_{a_n}$;

(iii) $\|\Sigma_{n=1}^{\infty}\alpha_n\epsilon_{a_n}\| = \Sigma_{n=1}^{\infty}|\alpha_n|$

and every non zero measure in $M_d(E)$ has a unique representation (i) *in which all α_n's are non zero (the sum may be finite).*

The extreme points of B_X, where $X = M_d(E)$, are given by

PROPOSITION 3.2. $\operatorname{ext}(B_X) = \{\sigma\epsilon_a : \sigma \in \mathbb{C}$ with $|\sigma| = 1$ and $a \in E\}$.

Proof. Let ν be a non zero measure in B_X and let $\nu = \Sigma_{n=1}^{\infty}\alpha_n\epsilon_{a_n}$ the representation (i) of ν. We assume that there exists an integer $k \geqslant 1$ such that

$|\alpha_k| \neq 1$. Then one can write $\nu = \lambda\mu_1 + (1-\lambda)\mu_2$, where

$$\lambda = |\alpha_k|\,, \quad \mu_1 = (\alpha_k/|\alpha_k|)\,\epsilon_{a_k} \quad \text{and} \quad \mu_2 = \Sigma_{n\neq k}\,\alpha_n(1-|\alpha_k|)^{-1}\epsilon_{a_n}\,.$$

We have $\lambda \in]0,1[$, $\mu_1,\mu_2 \in B_X$ and $\mu_1 \neq \mu_2$. Then $\nu \notin \mathrm{ext}(B_X)$. Conversely, it is easy to see that $\epsilon_a \in \mathrm{ext}(B_X)$ for all $a \in E$. ∎

Remark 3.3. The decomposition $\nu = \lambda\mu_1 + (1-\lambda)\mu_2$ given in the last proof, shows that $M_d(E)$ has the λ-property.

THEOREM 3.4. *Let E be an infinite locally compact space (non countable). Then $M_d(E)$ has the λ-property. In fact, if $\nu \in M_d(E)$ with $\|\nu\| \leqslant 1$ and $M = \sup\{|\nu(t)| : t \in E\}$, then $\lambda(\nu) = {}^1\!/_2(1-\|\nu\|+2M)$. Moreover, $\lambda(\nu)$ is attained.*

Proof. Let $\nu \in B_X \setminus \{0\}$. We assume that $\nu \notin \mathrm{ext}(B_X)$. Let $(e = \sigma\epsilon_a,\ \mu = \Sigma_{n=1}^{\infty}\beta_n\epsilon_{b_n},\ \lambda)$ a triple that is amenable to $\nu = \Sigma_{n=1}^{\infty}\alpha_n\epsilon_{a_n}$.

If $a \in \{a_j : j \geqslant 1\}$ and $a \notin \{b_j : j \geqslant 1\}$, then $\nu(a) = \lambda\sigma$ and $\lambda = |\nu(a)| \leqslant \sup\{|\nu(t)| : t \in E\} \leqslant {}^1\!/_2(1-\|\nu\|+2M)$.

If $a \in \{a_j : j \geqslant 1\} \cap \{b_j : j \geqslant 1\}$, one can assume that $a = a_1 = b_1$. Then we have $\alpha_1 = \lambda\sigma + (1-\lambda)\beta_1$ and there exists a bijection Θ from $\{2,3,\dots\}$ onto $\{2,3,\dots\}$ such that

$$\left.\begin{aligned} \alpha_n &= (1-\lambda)\beta_{\Theta(n)} \\ a_n &= b_{\Theta(n)} \end{aligned}\right\} \qquad \text{for } n = 2,3,\dots\,.$$

Hence

$$1 \geqslant \|\mu\| = \Sigma_{n=1}^{\infty}|\beta_n| = |\alpha_1 - \lambda\sigma|(1-\lambda)^{-1} + \Sigma_{n=2}^{\infty}|\alpha_n|(1-\lambda)^{-1} =$$

$$= (\|\nu\| - |\alpha_1| + |\lambda\sigma - \alpha_1|)(1-\lambda)^{-1} \geqslant (\|\nu\| - 2|\alpha_1| + \lambda)(1-\lambda)^{-1}\,.$$

Then $\lambda \leqslant {}^1\!/_2(1-\|\nu\|+2|\alpha_1|) \leqslant {}^1\!/_2(1-\|\nu\|+2M)$.

If $a \notin \{a_j : j \geqslant 1\}$, then $a \in \{b_j : j \geqslant 1\}$ and we can assume that $a = b_1$. Then we have $\lambda\sigma + (1-\lambda)\beta_1 = 0$ and

$$1 \geqslant \|\mu\| = |\beta_1| + \Sigma_{n=2}^{\infty}|\beta_n| = |\lambda\sigma|(1-\lambda)^{-1} + \Sigma_{n=1}^{\infty}|\alpha_n|(1-\lambda)^{-1} =$$

$$= (\|\nu\| + \lambda)(1-\lambda)^{-1}\,.$$

Hence $\lambda \leqslant {}^1\!/_2(1-\|\nu\|) \leqslant {}^1\!/_2(1-\|\nu\|+2M)$.

Conversely, let $k \geqslant 1$ such that $|\alpha_k| = |\nu(a_k)| = \sup\{|\nu(t)| : t \in T\}$. Let

$\lambda = {}^1/_2(1 - \|\nu\| + 2M)$, $e = \alpha_k M^{-1} \epsilon_{a_k}$ and $\mu = {}^1/_2(1-\lambda)^{-1}(\|\nu\| - 1)\, e + \Sigma_{n \neq k} \alpha_n (1-\lambda)^{-1} \epsilon_{a_n}$. We have $\|\mu\| = 1$ and $\nu = \lambda e + (1-\lambda)\mu$. ∎

Remark. Let $\{a_n\}_{n=1}^{\infty}$ a sequence of distinct points of E and $\nu_n = n^{-1} \Sigma_{j=1}^{n} \epsilon_{a_j}$ then $\lambda(\nu_n) = n^{-1}$; this shows that the space $M_d(E)$ has not the uniform λ–property.

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