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Note on some Integral Volterra Equations [1]

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1. INTRODUCTION

The nonlinear Volterra equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,\mathrm{d}s\,, \quad x \geqslant 0\,, \tag{1.1}$$

has been studied recently with connection to some problems in nonlinear diffusion and shock-wave propagation. In these problems the kernel k is nonnegative and g is an increasing continuous function such that $g(0)=0$. Obviously, $u \equiv 0$ is the trivial solution to (1.1). From a physical point of view, however, it is especially interesting to know when (1.1) has a nontrivial solution, i.e., continuous function u such that $u(x) > 0$ for $x > 0$. During the last few years some papers concerning the existence of nontrivial solutions have been written (see the list of references). All those papers have as a background Gripenberg's paper [9]. In that paper Gripenberg generalized the famous Osgood condition for integral equation of the type (1.1). This integral condition can be applied to wider classes of kernels k and nonlinearities g ([5], [6], [7], [9], [10], [11], [12], [13], [14], [15]). However there are conditions for the existence of nontrivial solutions which have not got an integral form ([8], [16], [17], [19]). In [16] they have the form of function series. To obtain these conditions it is necessary to assume that the kernel k is an integrable function such that $k > 0$ a.e. and g is a strictly increasing absolutely continuous function such that $g(0)=0$ and $u/g(u) \to 0$ as $u \to 0+$.

In this paper we want to show that conditions presented in [16] can be generalized to wider classes of kernels k and nonlinearities g.

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2. STATEMENT OF RESULTS

We shall study equation (1.1) assuming that

(k) $k:[0,\delta]\longrightarrow[0,+\infty]$, $\delta>0$, is an integrable function such that $K(x)>0$ for $x>0$, where $K(x)\equiv\int_0^x k(s)\,\mathrm{d}s$.

(g) $g:[0,+\infty)\longrightarrow[0,+\infty)$ is a nondecreasing continuous function such that $g(0)=0$, $g(x)>0$ for $x>0$ and $u/g(u)\to 0$ as $u\to 0+$.

If f is a continuous nondecreasing continuous function then we can define

$$f_{\mathrm{l}}^{-1}(y)\equiv\min\{x:f(x)=y\}\quad\text{and}\quad f_{\mathrm{r}}^{-1}(y)\equiv\max\{x:f(x)=y\}\,.$$

For a given function h we define the sequence of functions h^n, $n=0,1,\dots$, as follows: $h^0(x)=x$, $h^{n+1}=h^n\circ h$, $n=0,1,\dots$.

We formulate the following necessary condition

THEOREM 2.1. *Let* (k) *and* (g) *be satisfied. Let* ψ *be a continuous function such that* $\psi(x)>0$ *for* $x>0$ *and* $\overline{\lim}_{x\to0+}\{g(x)/\psi(x)\}<1$. *If equation* (1.1) *has a nontrivial strictly increasing solution on an interval, then the series*

$$\sum_{n=0}^{\infty}K_{\mathrm{l}}^{-1}\Big((g_{\mathrm{r}}^{-1})^n(x)\Big/\psi((g_{\mathrm{r}}^{-1})^n(x))\Big)\tag{2.1}$$

is convergent on $[0,\delta_0]$, $\delta_0>0$.

Moreover, the following sufficient condition holds.

THEOREM 2.2. *Let* (k) *and* (g) *be satisfied. Let* ϕ *be a continuous nondecreasing function on* $[0,\delta_0]$, $\delta_0>0$, *such that* $0<\phi(x)<g(x)$ *for* $x\in(0,\delta_0]$ *and* $x/\phi(x)\to0$ as $u\to0+$. *If the series*

$$\sum_{n=0}^{\infty}K_{\mathrm{l}}^{-1}\Big((g_{\mathrm{l}}^{-1}\circ\phi)^n(x)\Big/\phi((g_{\mathrm{l}}^{-1}\circ\phi)^n(x))\Big)\tag{2.2}$$

is convergent at $x_0\in(0,\delta_0]$, *then equation* (1.1) *has a nontrivial solution on some interval.*

3. SOME PRELIMINARIES

On the basis of results presented in [5] we know:

PROPOSITION 3.1. *If equation* (1.1) *has a nontrivial solution then it is nondecreasing function.*

An integration by parts gives the following:

PROPOSITION 3.2. *If u is a nontrivial solution to (1.1) then u is an absolutely continuous function such that*

$$u(x) = \int_0^x K(x-s)\,\mathrm{d}(g\circ u)(s)$$

on $[0,\delta]$, $\delta > 0$.

We can formulate the corollary.

COROLLARY 3.1. *Let* (k) *and* (g) *be satisfied. Let $k > 0$ a.e. or g is strictly increasing. If* (1.1) *has a nontrivial solution then u is strictly increasing.*

Remark 3.1. For every $\epsilon \in (0,1)$ the equation

$$u_\epsilon(x) = \epsilon x + \int_0^x k(x-s)\,g(u(s))\,\mathrm{d}s \tag{3.1}$$

has a unique strictly increasing absolutely continuous solution u_ϵ on an interval $[0,\delta_1]$, where $\delta_1 > 0$ is independent of ϵ. Moreover, $u_{\epsilon_1} \leqslant u_{\epsilon_2}$ for $\epsilon_1 \leqslant \epsilon_2$ (see [16]).

We can prove the following lemma:

LEMMA 3.1. *Let $\epsilon \in [0,1)$. If u_ϵ is the nontrivial strictly increasing solution to* (3.1) *then the inverse function u_ϵ^{-1} satisfies the equation*

$$x = \epsilon\, u_\epsilon^{-1}(x) + \int_0^x K(u_\epsilon^{-1}(x) - u_\epsilon^{-1}(s))\,\mathrm{d}g(s) \tag{3.2}$$

for $x \in [0, u_\epsilon^{-1}(\delta_1)]$.

Proof. Let $\epsilon \geqslant 0$ and u_ϵ be the solution to (3.1) mentioned in Corollary 3.1 and Remark 3.1. Since u_ϵ is absolutely continuous and strictly increasing then

$$u_\epsilon(x) = \epsilon x + \int_0^{u_\epsilon(x)} k(x - u_\epsilon^{-1}(s))(u_\epsilon^{-1}(s))'\,g(s))\,\mathrm{d}s\,.$$

Integrating by parts we obtain

$$u_\epsilon(x) = \epsilon x + \int_0^{u_\epsilon(x)} K(x - u_\epsilon^{-1}(s))\,\mathrm{d}g(s))\,.$$

Substituting $u_\epsilon^{-1}(x)$ for x gives (3.2). ∎

Remark 3.2. The function $G_\epsilon(x,s) \equiv K(u_\epsilon^{-1}(x) - u_\epsilon^{-1}(s))$ is decreasing with respect to s. Moreover, $G_\epsilon(x,0) \equiv K(u_\epsilon^{-1}(x))$ and $G_\epsilon(x,x) = 0$.

4. Proof of the Necessary Condition

First we prove the following lemma.

LEMMA 4.1. *Let* ψ *be a continuous function such that* $\psi(x) > 0$ *for* $x > 0$ *and* $\overline{\lim}_{x\to 0+}\{g(x)/\psi(x)\} < 1$. *If equation* (1.1) *has the nontrivial strictly increasing solution* u_0, *then*

$$u_0^{-1}(x) \geqslant K_1^{-1}(x/\psi(x)) + u_0^{-1}(g_r^{-1}(x)) \tag{4.1}$$

on an interval $[0, \delta_0]$, $\delta_0 > 0$.

Proof. Let us note that for the proof of (4.1) it is sufficient to show that

$$G_0(x, g_r^{-1}(x)) \geqslant x/\psi(x) \tag{4.2}$$

for $x \in [0, \delta_0]$ $(\delta_0 > 0)$. Suppose (4.2) does not hold. Hence there exists a sequence $x_n \to 0$ as $u \to \infty$ such that

$$G_0(x_n, g_r^{-1}(x_n)) < x_n/\psi(x_n)\,. \tag{4.3}$$

From (3.2) we obtain

$$x_n = \int_0^{g_r^{-1}(x_n)} G_0(x_n, s)\,dg(s) + \int_{g_r^{-1}(x_n)}^{x_n} G_0(x_n, s)\,dg(s)\,. \tag{4.4}$$

Since G_0 is decreasing with respect to s, we get

$$x_n \leqslant K(u_0^{-1}(x_n))\,g(g_r^{-1}(x_n)) + g(x_n)\,G_0(x_n, g_r^{-1}(x_n))\,. \tag{4.5}$$

By (4.3) and (4.5) we obtain

$$x_n \leqslant K(u_0^{-1}(x_n))\,x_n + x_n g(x_n)/\psi(x_n)\,.$$

From the last inequality we get

$$1 \leqslant K(u_0^{-1}(x_n)) + g(x_n)/\psi(x_n)\,. \tag{4.6}$$

Since $\overline{\lim}_{x\to 0+}\{g(x)/\psi(x)\} < 1$ then $g(x)/\psi(x) < 1 - \eta$ $(\eta \in (0,1))$ for $x > 0$ sufficiently small. Hence we have

$$1 \leqslant K(u_0^{-1}(x_n)) + 1 - \eta \tag{4.7}$$

for $n \geqslant n_0$. If $n \to \infty$ then $K(u_0^{-1}(x_n)) \to 0$. From (4.7) we get contradiction. ∎

Proof of Theorem 2.1. Let ψ satisfy the assumptions of the theorem. If u_0 is the strictly increasing nontrivial solution to (1.1) then inequality (4.1) holds.

We can iterate (4.1). After n iterations we get

$$(4.8)\quad u_0^{-1}(x) \geqslant \sum_{i=0}^{n} K_1^{-1}\big((g_r^{-1})^i(x)/\psi((g_r^{-1})^i(x))\big) + u_0^{-1}((g_r^{-1})^{n+1}(x))$$

on $[0,\delta_0]$. Without loss of generality we can assume $g_r^{-1}(x) < x$ on $(0,\delta_0]$. If $n \to \infty$ then from (4.8) we get (2.1). ∎

5. Proof of the Sufficient Condition

The following lemma is true:

LEMMA 5.1. *Let ϕ be a nondecreasing continuous function on $[0,\delta_0]$, $\delta_0 > 0$, such that $x < \phi(x) < g(x)$ for $x \in (0,\delta_0]$ and $x/\phi(x) \to 0$ as $x \to 0+$. Let $\epsilon > 0$. If u_ϵ is the solution to (3.1) then*

$$(5.1)\qquad u_\epsilon^{-1}(x) \leqslant K_r^{-1}(x/\phi(x)) + u_\epsilon^{-1}((g_1^{-1}\circ\phi)(x))$$

for $x \in [0, u_\epsilon(\delta_1)]$ $(u_\epsilon(\delta_1) < \delta_0$ for $\epsilon \in (0,1))$.

Proof. On the basis of (3.2) we get

$$(5.2)\qquad x \geqslant \int_0^{(g_1^{-1}\circ\phi)(x)} G_\epsilon(x,s)\,dg(s)\,.$$

By Remark 3.2 and properties of ϕ we obtain

$$(5.3)\qquad \int_0^{(g_1^{-1}\circ\phi)(x)} G_\epsilon(x,s)\,dg(s) \geqslant \phi(x)\,G_\epsilon(x,(g_1^{-1}\circ\phi)(x))\,.$$

Using (5.2) and (5.3) gives

$$(5.4)\qquad x/\phi(x) \geqslant G_\epsilon(x,(g_1^{-1}\circ\phi)(x))\,.$$

From (5.4) we obtain (5.1). ∎

Proof of Theorem 2.2. Let ϕ be given on $[0,\delta_0]$. Let $\{u_\epsilon : \epsilon \in (0,1)\}$ denote the family of solutions to (3.1) on $[0,\delta_1]$ mentioned in Remark 3.1. Fixing ϵ we can iterate the inequality (5.1). After n iterations we get

$$(5.5)\quad u_\epsilon^{-1}(x) \leqslant \sum_{i=0}^{n} K_r^{-1}\Big((g_1^{-1}\circ\phi)^i(x)\big/\phi((g_1^{-1}\circ\phi)^i(x))\Big) + u_\epsilon^{-1}((g_1^{-1}\circ\phi)^{n+1}(x))$$

on $[0, u_\epsilon(\delta_1)]$. Since $(g_1^{-1}\circ\phi)(x) < x$ for $x > 0$ then using similar arguments as in [16] we get

(5.6) $$u_\epsilon^{-1}(x) \leqslant F(x)$$

on $[0, u_\epsilon(\delta_1)]$, where

$$F(x) \equiv \sum_{i=0}^{\infty} K_r^{-1}\Big((g_1^{-1}\circ\phi)^n(x)/\phi(g_1^{-1}\circ\phi)^n(x))\Big)$$

is well defined for $x \leqslant x_0$ (see [17]). Let us note that $\lim_{x\to 0+} F(x) = 0$. We can find a strictly increasing continuous function $\overline{F}$ such that $F \leqslant \overline{F}$. Hence we get

(5.7) $$u_\epsilon(x) \leqslant \overline{F}^{-1}(x)$$

for $x \in [0, \delta_1]$, where $\overline{F}^{-1}$ is the inverse function to $\overline{F}$. If $\epsilon \to 0+$ then the sequence u_ϵ tends to nontrivial solution u of (1.1) because $u \geqslant \overline{F}^{-1}$ on $[0, \delta_1]$ (for details see [16]). The theorem is proved. ∎

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