EXTRACTA MATHEMATICAE Vol. 8, Núm. 1, 49 – 53 (1993)

On the Derivation of a First–Order Canonical Set of Hyperbolic Delaunay–type Elements [1,2]

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AMS *Subject Class.* (1991): 70H15, 70F05, 58F05. *Received January* 21, 1993

1. INTRODUCTION

This *Note* tackles the investigation of a *symplectic transformation* in the 6–dimensional phase space that allows one to reduce a perturbed positive–energy two–body problem into a pure Keplerian motion along a hyperbolic orbit formulated in a suitable chart. As a result of this, a canonical set of phase variables is obtained that is applicable to the study of hyperbolic–type orbital motion and contains the perturbation emanating from the considered Deprit potential as an effect incorporated into its own definition.

The potential covered in the present research constitutes a simplified mathematical model for the major perturbation due to the *oblateness* of a spheroid under whose gravitational attraction the motion of a natural or artificial orbiter occurs. To be precise, the potential is that related to the so–called *radial intermediary* of the first order introduced, after application of Lie transform techniques, by *Deprit* ([1], p. 138) as an integrable Hamiltonian approximation to the *Main Problem in Satellite Theory*. To formalize these comments in concrete terms, the canonical set of *Hill–Whittaker polar nodal variables* $(r,\theta,\nu\,;p_r,p_\theta,p_\nu)$ coordinatizes the 6–dimensional phase space. In this symplectic chart, the canonical formulation of the *Main Problem* of the theory of motion of *zonal satellites* leads to an investigation of the dynamical system governed by

[1] Partial financial support has been contributed by CICYT of Spain (Project ESP.88–0541) and by University of Valladolid (Programme 541A, MGV).

[2] KEY WORDS: hyperbolic two–body problem, Delaunay canonical elements, perturbed Keplerian motion, planetary oblateness, zonal satellite, Deprit intermediary.

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$$\mathcal{M} = \mathcal{H}_0(r,-,-;p_r,p_\theta,-) \; + \; \epsilon\,\mathcal{M}_1(r,\theta,-;p_r,p_\theta,p_\nu) =$$

$$= \tfrac{1}{2}\{p_r^2 + (p_\theta^2/r^2)\} \; - \; \mu/r \; + \; \{(\epsilon\mu R_e^2)/(4r^3)\}(3c^2 - 1 + 3s^2\cos 2\theta)\,,$$

with the customary symbols for the functions of the inclination $I \equiv I(p_\theta, p_\nu)$: $c \equiv \cos I = p_\nu/p_\theta$, $s \equiv \sin I$; on the other hand, the Hamiltonian characterizing the *radial intermediary* here taken into account can be assumed to be of the form

$$\mathcal{H} \equiv \mathcal{H}(r,-,-;p_r,p_\theta,p_\nu) = \mathcal{H}_0 \; + \; \{(\epsilon\mu^2 R_e^2)/(4r^2 p_\theta^2)\}(3c^2 - 1)\,.$$

As for the notations, $\mathcal{H}_0$ represents the Hamiltonian of an unperturbed Kepler problem, R_e stands for the mean equatorial radius of the central body, and the (small) parameter $\epsilon = -J_2$ is a *dimensionless measure of the flattening of the primary*.

2. A Hyperbolic–Like Delaunay–Levi-Cività Mapping

A canonical map will be defined in the phase space via a generating function inspired by the previously *fixed* Deprit Hamiltonian. Conditions are assumed such that the dynamical problem to which the transformation is to be applied will result in orbital motion with *positive total energy*. Thus, the intermediary will then be brought into (hyperbolic) Keplerian form with the help of the proposed symplectic mapping.

Anyway, in spite of this approach and these aims, it should be borne in mind that (as clearly stated in [1], p. 115) *the transformation stands on its own as a symplectic change of variables in the phase space*, *irrespective of the dynamical system to which it may be applied* and belongs to the Geometry of the phase space more than to the Dynamics.

The following development generalizes what Deprit ([1], pp. 115–118 and 124–126) calls a *Delaunay transformation* or a *Delaunay map*, which (in the light of [5]) can also be named after *Levi-Cività*. As for a derivation of the elliptic Delaunay set from that of Hill-Whittaker, the above reference to [1] will suffice for the purposes of this *Note*. The *generalization* must be understood along *two main lines*: **(a)** *hyperbolic–type* motion is considered; **(b)** certain *perturbations* of Kepler problems are taken into account.

In the 6–dimensional phase space one will carry out the canonical transformation $(r,\theta,\nu;p_r,p_\theta,p_\nu) \longrightarrow (l,g,h;L,G,H)$, derived from the generating function

$$S \equiv S(r,\theta,\nu;L,G,H) = \theta G + \nu H + \int_{r_0}^{r} \sqrt{Q}\, dr ,$$

where

$$Q \equiv Q(r;L,G,H) =$$

$$= 2\mu/r + \mu^2/L^2 - G^2/r^2 - \{(2\epsilon\mu^2 R_e^2)/(4r^2G^2)\}(3H^2/G^2 - 1) .$$

The lower limit $r_0 \equiv r_0(L,G,H)$ in the integral is the only positive root of the r–equation $Q(r;L,G,H) = 0$. The *generating relations of the transformation* yield

$$L = -\sqrt{\mu a} , \qquad l = e \sinh F - F ,$$

$$G = p_\theta , \qquad g = \theta - \Delta_{(\theta)} f ,$$

$$H = p_\nu , \qquad h = \nu - \Delta_{(\nu)} f ,$$

with the following meaning for the subsidiary quantities and auxiliary variables

$$L = -\sqrt{\mu a} , \quad \mu a(e^2 - 1) = G^2 + \{(\epsilon\mu^2 R_e^2)/(2G^2)\}(3c^2 - 1) \equiv \Gamma^2 ,$$

$$p = \Gamma^2/\mu = a(e^2 - 1) , \quad e^2 = 1 + \Gamma^2/L^2 > 1 ,$$

$$(a + r)/ae = \cosh F , \quad r = a(e \cosh F - 1) , \quad r = p/(1 + e \cos f) ,$$

$$\Delta_{(\theta)} = \{G + [(\epsilon\mu^2 R_e^2)/(2G^3)](1 - 6c^2)\}/\Gamma , \; \Delta_{(\nu)} = \{(\epsilon 3\mu^2 R_e^2\, c)/(2G^3)\}/\Gamma .$$

For future use in perturbation studies, it is stressed that the important θ–equation now reads $\theta = g + \Delta_{(\theta)} f$, with a contribution of the order of ϵ due to the terms in the partial derivative of the Deprit potential with respect to G.

The application of the transformation to $\mathcal{H}$ gives a simpler form of this function, easily recognizable as formulating a pure (hyperbolic) Kepler motion in the new chart. Observe that no change of time parameter is invoked: the physical time acts as the independent variable in terms of which the canonical equations derived from the *reduced* Hamiltonian

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(-,-,-;L,-,-) = \mu^2/2L^2$$

are integrated in this chart and yield a *hyperbolic Keplerian-like solution* to the system governed by $\mathcal{H}$: the variables g, h, L, G and H are constants of the motion, while

$$\frac{dl}{dt} = \frac{\partial \tilde{\mathcal{K}}}{\partial L} = -\mu^2/L^3 = \sqrt{\mu/a^3} = n \quad \Longrightarrow \quad l = nt + \text{const.}\,,$$

using $\mu = n^2 a^3$ to introduce a quantity n of the kind of a (hyperbolic) *mean motion*.

Thus, in view of the transformation equations, after solving for the original polar nodal variables in terms of the auxiliary integration variables F and f and taking advantage of the subsidiary quantities previously introduced, the *solution to the intermediary* problem posed by $\mathcal{K}$ may be schematized by giving a *parametric* representation

$$p_r = \sqrt{\mu/a}\,(e\sinh F)/(e\cosh F - 1) = \sqrt{\mu/p}\; e\sin f\,,$$

$$r = a(e\cosh F - 1) = p/(1 + e\cos f)\,,$$

$$p_\theta = G = \text{const.}\,, \qquad \theta = g + \Delta_{(\theta)} f\,,$$

$$p_\nu = H = \text{const.}\,, \qquad \nu = h + \Delta_{(\nu)} f\,,$$

together with the (*hyperbolic*) *Kepler equation*: $l = e\sinh F - F = nt + \text{const.}$

The auxiliary variables F and f can be regarded as the hyperbolic eccentric and true anomalies for *a fictitious Keplerian motion* with semi-latus rectum, semi-major axis, eccentricity, mean motion and angular momentum magnitude given by the quantities p, a, e, n and Γ. In this approach, the root r_0 of the r-equation $Q = 0$ (namely, $r_0 = a(e-1)$) represents the *periapsis* for this hypothetic motion. Thus, the orbiter might be viewed as *simulating a Keplerian orbit* whose main geometrical and dynamical features related to *form* and *size* are described by the above elements. This interpretation extends Deprit's analysis ([1]) concerning *quasi-Keplerian systems*. However, in contrast to Deprit's study, the present research is devoted to hyperbolic-type motion under the action of the chosen intermediary Hamiltonian.

Remember that by neglecting the perturbation term in the generating function S, the resulting canonical mapping effects the transition from the Hill-Whittaker chart to the set of (hyperbolic) canonical Delaunay elements presented in [4] and recovered in [2].

FINAL REMARKS

(1) Under the effect of the symplectic mapping generated by S, *the non-trigonometric terms of the first order are removed from the Hamiltonian of the*

Main Problem; in the positive-energy case this result is analogous to the elimination of the first-order secular part accomplished by the corresponding transformation when bounded states of motion under the same potential are investigated. Concerning this point within the framework of a generalized-DS-approach to satellite motion, the reader is referred to previous results, e.g., the paper [3].

(2) In order to arrive at an *approximate analytical solution to the positive-energy Main Problem*, relying on a modification of the von Zeipel method as presented in [4], a consistent application of the basic perturbation technique, (along with the idea of imposing conditions at $r = \infty$), allows one to obtain the generating function for a *near-identity canonical transformation* that allows one to evaluate the perturbation effects due to the primary's flattening. This is a rather long reckoning task whose results will be presented in a forthcoming paper dealing with the close-to-perigee orbital motion of non-recurrent celestial bodies acted by the major terms of the gravitational potential stemming from an oblate spheroid considered as the central attracting body.

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