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On a Type of P–Sasakian Manifold

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INTRODUCTION.

Let (M,g) be an n–dimensional Riemannian manifold admitting a 1–form η which satisfies the conditions

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0 \tag{1}$$

$$(\nabla_X \nabla_Y \eta)(z) = -g(X,Z)\,\eta(Y) - g(X,Y)\,\eta(Z) + 2\eta(X)\,\eta(Y)\,\eta(Z) \tag{2}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M,g) admits a vector field ξ and a (1–1) tensor field ϕ such that

$$g(X,\xi) = \eta(X) \tag{3}$$

$$\eta(\xi) = 1 \tag{4}$$

$$\nabla_X \xi = \phi X \tag{5}$$

then such a manifold is called a Para–Sasakian manifold or briefly a P–Sasakian manifold by T. Adati and K. Matsumoto [1] which are considered as special cases of an almost paracontact manifold introduced by I. Sato [3].

In this paper we study some remarkable properties of P–Sasakian manifolds which satisfy some conditions on the Ricci tensor or the Riemannian curvature tensor, for example, $R(X,Y).R=0$ or $R(X,Y).S=0$ where R and S are the Riemannian curvature tensor and the Ricci tensor respectively and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In the last section we consider P–Sasakian manifolds which are hypersurfaces of a Riemannian manifold of constant curvature 1 and a necessary and sufficient for a P–Sasakian manifold to be minimal has been obtained.

1. PRELIMINARIES. It is known [1],[3] that in a P–Sasakian manifold the following relations hold:

$$\phi\xi = 0 \tag{1.1}$$

$$\phi^2 X = X - \eta(X)\xi \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.3}$$

$$S(X,\xi) = -(n-1)\eta(X) \tag{1.4}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \tag{1.5}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \tag{1.6}$$

$$R(\xi,X)\xi = X - \eta(X)\xi \tag{1.7}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{1.8}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{1.9}$$

The above results will be used in the next section.

2. SOME CONDITIONS ON THE RICCI TENSOR AND THE RIEMANNIAN CURVATURE TENSOR IN A P–SASAKIAN MANIFOLD.

THEOREM 1. *Let M be a P–Sasakian manifold. Then the following conditions are equivalent:*

i) M is an Einstein manifold.

ii) The Ricci tensor is parallel, $\nabla S = 0$.

iii) $R(X,Y).S = 0$ for any X and Y.

Proof. *i)* $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* is clear. Assume condition *iii)*, which is equivalent to

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0 \tag{2.1}$$

for any vectors U and V. Putting $X=\xi$ in (2.1) we get

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{2.2}$$

Using (1.4) and (1.6) we get from (2.2)

$$\begin{aligned} &\eta(U)S(Y,V) + (n-1)\eta(V)g(Y,U) + \\ &S(U,Y)\eta(V) + (n-1)g(Y,V)\eta(U) = 0 \end{aligned} \tag{2.3}$$

Now putting $U=\xi$ in (2.3) we get

(2.4) $$S(Y,V)=-(n-1)g(Y,V).$$

Therefore M is an Einstein manifold. This completes the proof. ∎

If the Ricci curvature tensor S is of the form

(2.5) $$S=ag+b\eta\otimes\eta$$

where a and b are functions on M, then M is called an η-Einstein manifold. Now it is known that

(2.6) $$(R(X,Y).S)(U,V)=-S(R(X,Y)U,V)-S(U,R(X,Y)V).$$

Using (1.5) and (2.5) we get from (2.6)

(2.7) $$\begin{aligned}(R(X,Y).S)(U,V)=b[\eta(V)g(Y,U)+\eta(U)g(Y,V)]\,\eta(X)-\\ b[\eta(V)g(X,U)+\eta(U)g(X,V)]\,\eta(Y)\end{aligned}$$

Assume that (2.7) holds. Then for $Y=\xi$ we get

(2.8) $$(R(X,\xi).S)(U,V)=b[2\eta(U)\eta(V)\eta(X)-\eta(V)g(X,U)-\eta(U)g(X,V)].$$

Now as in the proof of the Theorem 1 we get from (2.8)

$$S(U,X)+(n-1)g(X,U)=b[\eta(U)\eta(X)-g(X,U)]$$

or,

$$S(U,X)=b\,\eta(U)\eta(X)-[b+(n-1)]g(X,U)$$

which shows that M is an η-Einstein manifold. Thus we can state the following theorem:

THEOREM 2. *A P-Sasakian manifold is an η-Einstein manifold iff* (2.7) *holds.*

THEOREM 3. *Let M be a P-Sasakian manifold. Then the following conditions are equivalent:*

i) M is of constant curvature.

ii) M is locally symmetric $\nabla R=0$.

iii) $R(X,Y).R=0$ for any X and Y.

Proof. *i)* $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* is clear. Assume conditions *iii)* which is equivalent to

(3.9) $$\begin{aligned}R(X,Y)R(U,V)W-R(R(X,Y)U,V)W-\\ R(U,R(X,Y)V)W-R(U,V)R(X,Y)W=0.\end{aligned}$$

Therefore,

$$g[R(\xi,Y)R(U,V)W,\xi]-g[R(R(\xi,Y)U,V)W,\xi]-$$
$$g[R(U,R(\xi,Y)V)W,\xi]-g[R(U,V)R(\xi,Y)W,\xi]=0 \tag{3.10}$$

Using (2.6) we get from (3.10)

$$'R(U,V,W,Y)-\eta(Y)\eta(R(U,V)W)+\eta(U)\eta(R(Y,V)W)+$$
$$\eta(V)\eta(R(U,Y)W)+\eta(W)\eta(R(U,V)Y)- \tag{3.11}$$
$$g(Y,U)\eta(R(\xi,V)W)-g(Y,V)\eta(R(U,\xi)W)-g(Y,W)\eta(R(U,V)\xi)=0$$

where

$$'R(U,V,W,Y)=g(R(U,V)W,Y).$$

Putting $Y=U$ in (3.11) we get

$$'R(U,V,W,U)+\eta(V)\eta(R(U,U)W)+\eta(W)\eta(R(U,V)U)-$$
$$g(U,U)\eta(R(\xi,V)W)-g(U,V)\eta(R(U,\xi)W)-g(U,W)\eta(R(U,V)\xi)=0. \tag{3.12}$$

Let $\{e_i\}$, $i=1,2,...,n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1\leqslant i\leqslant n$ of the relation (3.12) for $U=e_i$ gives

$$\eta(R(\xi,V)W)=\eta(V)\eta(W)+\frac{1}{n-1}S(V,W). \tag{3.13}$$

Now from (1.3), (1.4) and (2.5) we have

$$\eta(R(\xi,V)W)=\eta(V)\eta(W)-g(V,W). \tag{3.14}$$

From (3.13) and (3.14) we get

$$S(V,W)=-(n-1)g(V,W). \tag{3.15}$$

Using (1.3) and (2.5) it follows from (3.11) that

$$'R(U,V,W,Y)-\eta(Y)\eta(R(U,V)W)+\eta(U)\eta(R(Y,V)W)+$$
$$\eta(V)\eta(R(U,Y)W)+\eta(W)\eta(R(U,V)Y)- \tag{3.16}$$
$$g(Y,U)\eta(R(\xi,V)W)-g(Y,V)\eta(R(U,\xi)W)=0$$

Using (2.5) and (3.15) it follows from (3.11) that

$$'R(U,V,W,Y)=-[g(Y,U)g(V,W)-g(Y,V)g(U,W)].$$

Thus the manifold is of constant curvature. This completes the proof. ∎

Remark. In Theorem 3 *iii)* $\Rightarrow$ *i)* was given by T. Takahashi [4] for a Sasakian manifold in another way.

3. P–SASAKIAN MANIFOLDS WHICH ARE HYPERSURFACES OF A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE 1.

Let M be a P–Sasakian manifold of dimension n which is isometrically immersed in a Riemannian manifold N of dimension $(n+1)$ of constant curvature 1. The we have the Gauss and Codazzi equations [2]

$$R(X,Y)=X\wedge Y+AX\wedge AY \tag{3.1}$$

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y+g(AY,Z)AX-g(AX,Z)AY \tag{3.2}$$

$$(\nabla_X A)(Y)=(\nabla_Y A)(X) \tag{3.3}$$

where A is a $(1-1)$ tensor field associated with the second fundamental form B by $B(X,Y)=g(X,AY)$. A is symmetric with respect to g and when the trace of A vanishes, M or the immersion is called minimal. The rank of A is called the type number of the immersion.

Since the Ricci curvature tensor S is given by

$$S(X,Y)=\text{trace}[\,U\longrightarrow R(X,U)Y]$$

by (3.2) we have

$$S(X,Y)=(n-1)g(X,Y)+(\text{trace } A)g(AX,Y)-g(AAX,Y). \tag{3.4}$$

Replacing X and Y by ϕX and ϕY in (3.4) we have

$$S(\phi X,\phi Y)=(n-1)g(\phi X,\phi Y)+\theta g(A\phi X,\phi Y)-g(AA\phi X,\phi Y) \tag{3.5}$$

where θ is the trace of A.

Since ϕ is symmetric we have

$$g(A\phi X,\phi Y)=g(\phi A\phi X,Y)$$

$$g(AA\phi X,\phi Y)=g(\phi AA\phi X,Y).$$

Using (1.3) and (1.9) it follows from (3.5)

$$\begin{aligned}S(X,Y)=(n-1)g(X,Y)-2(n-1)\eta(X)\eta(Y)+\\ \theta g(\phi A\phi X,Y)-g(\phi AA\phi X,Y).\end{aligned} \tag{3.6}$$

Then (1.3), (3.4) and (3.6) imply

$$2(n-1)(I-\phi^2)-\theta A-AA-\theta\phi A\phi+\phi AA\phi=0. \tag{3.7}$$

From (3.6) the scalar curvature $r=\text{trace } S$ is given by

$$r = -n(n-1) + \theta^2 - \text{trace } AA.$$

Now if trace of A vanishes then from (3.7) we get

$$2(n-1)(I-\phi^2) - AA + \phi AA\phi = 0. \tag{3.8}$$

Again if (3.8) holds then we have from (3.7) $\theta = 0$. Hence we can state the following theorems:

THEOREM 4. *A necessary and sufficient condition for a P–Sasakian manifold to be minimal is that* (3.8) *holds.*

THEOREM 5. *If a P–Sasakian manifold is isometrically immersed in a Riemannian manifold of constant curvature* 1 *then the scalar curvature is given by*

$$r = -n(n-1) + \theta^2 - \text{trace } AA$$

where θ is the trace of A.

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