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Coexistence States for Nonlinear Elliptic Problems Arising from Biology

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In this work we present some of the main results obtained in [1], [2], [3], related to the existence of nonnegative and nontrivial solutions of nonlinear elliptic systems arising from population dynamics.

It is well known that the evolution on the time of two interacting species u, v, living in a bounded habitat has been modelled by nonlinear reaction –diffusion systems whose equilibrium states (i.e., solutions which are independent of the time) originate the study of problems of the form

$$
\begin{aligned}
-\Delta u(x) &= u(x)\,(a(x)-b(x)u(x)+c(x)v(x))\,, \quad x\in\Omega\,,\\
-\Delta v(x) &= v(x)\,(e(x)-f(x)v(x)+g(x)u(x))\,, \quad x\in\Omega\,,\\
u(x) &= v(x)=0\,, \quad x\in\partial\Omega\,,
\end{aligned}
\tag{1}
$$

where Ω is a bounded and regular domain in $\mathbb{R}^n$, Δ is the Laplacian operator and a, b, c, e, f, g are real lipschitzian functions defined on $\overline{\Omega}$ with b and f strictly positive in $\overline{\Omega}$. In (1), the boundary condition may be interpreted as that the species may not stay on $\partial\Omega$ (the boundary of Ω). In the corresponding time–dependent problem, the terms $a(x)u(x)-b(x)u^2(x)$ and $e(x)v(x)-f(x)v^2(x)$ give the rate of increase of the species u and v respectively whereas the terms $c(x)u(x)v(x)$ and $g(x)u(x)v(x)$ show the interaction between the species.

Because of the origin of problem (1), we are only interested in the existence of nonnegative and nontrivial solutions (u, v) of (1) (solutions of (1) are understood in a classical sense, i.e., u and v must be functions of class $\mathscr{C}^2$ in Ω and continuous in $\overline{\Omega}$). More precisely, a coexistence state for (1) is a solution (u, v) such that both components are nonnegative and nontrivial.

From the early work by Volterra at the begining of this century many authors have been interested in the problem. Basically, sufficient (and sometimes necessary) conditions to obtain coexistence states for (1) have been given attending to three standard types of models: competition model (when the functions c and g are both negative in $\overline{\Omega}$), prey–predator model (c is, for example, negative and g positive in $\overline{\Omega}$) and cooperative model (c and g are both positive in $\overline{\Omega}$) (see [5], [6], [7], [8]). However, in the applications, it is usual to find problems which are none of the mentioned types. This may be produced by different causes. For instance, sometimes the function c or g or both have not a constant sign on $\overline{\Omega}$ (because of (1) may be of prey–predator type in $\Omega_1 \subset \Omega$ and of competition type in $\Omega \setminus \Omega_1$ as in the example bellow). Other times the type of interaction expressed in (1) is not the adequate for the model and this gives rise to a "nonlinear interaction" (see [4]) ,so that we must replace (1) by a more general model.

The previous considerations motivate the study of problems (which include (1) as a particular case) like

$$\begin{aligned} -\Delta u(x) &= u(x)\, f(x, u(x), v(x)), \quad x \in \Omega, \\ -\Delta v(x) &= v(x)\, g(x, u(x), v(x)), \quad x \in \Omega, \\ u(x) &= v(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $f, g : \overline{\Omega} \times [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ are lipschitzian functions.

To prove our main result we shall need the following hypotheses:

(H1) f is strictly decreasing with respect to u and g is strictly decreasing with respect to v.

(H2) There exist $\alpha > 0$, $\beta > 0$ such that

$$f(x, \alpha, s) \leqslant 0 \quad , \quad g(x, s, \beta) \leqslant 0 \ , \quad \forall x \in \overline{\Omega} \ , \ \forall s \geqslant 0.$$

Moreover, $u(0)$ and $v(0)$ mean, respectively, the maximal nonnegative solutions of the problems:

$$\begin{aligned} -\Delta u(x) &= u(x)\, f(x, u(x), 0), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta v(x) &= v(x)\, g(x, 0, v(x)), \quad x \in \Omega, \\ v(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Now, we may enunciate the following Theorem.

THEOREM 1 ([1]). *Let us suppose that* f *and* g *satisfy* (H1) *and* (H2). *Then, a sufficient condition to obtain coexistence for* (2) *is*

$$\lambda_1(\Omega,-f(.,0,v(0)(.)))<0 \quad \text{and} \quad \lambda_1(\Omega,-g(.,u(0)(.),0))<0 \tag{3}$$

where for $q \in \mathscr{C}(\overline{\Omega})$ *(continuous functions on* $\overline{\Omega}$*),* $\lambda_1(\Omega,q)$ *denotes the principal eigenvalue of the eigenvalue problem*

$$-\Delta u(x) + q(x)u(x) = \lambda u(x)\,, \qquad x \in \Omega\,,$$
$$u(x) = 0\,, \quad x \in \partial\Omega\,,$$

Outline of the proof. It combines a decoupling method to transform (2) into a single equation together with the use of a global bifurcation result.

For $v \in \mathfrak{L}(\overline{\Omega})$, (lipschitzian functions defined on $\overline{\Omega}$), let $u(v)$ the maximal nonnegative solution of the problem

$$\begin{aligned} -\Delta u(x) &= u(x)\, f(x,u(x),v(x))\,, \quad x \in \Omega\,,\\ u(x) &= 0\,, \quad x \in \partial\Omega\,. \end{aligned} \tag{4}$$

Let us consider the problem

$$\begin{aligned} -\Delta v(x) &= v(x)\; g(x,u(|v|)(x),|v(x)|)\,, \quad x \in \Omega\,,\\ v(x) &= 0\,, \quad x \in \partial\Omega\,. \end{aligned} \tag{5}$$

Then (2) has a coexistence state if and only if (5) has a nonnegative and nontrivial solution v such that $u(v) \not\equiv 0$. To find that v, we consider the family of problems

$$\begin{aligned} &-\Delta v(x) + (-g(x,u(0)(x),0))v(x) =\\ &= \lambda v(x) + v(x)[g(x,u(|v|)(x),|v(x)|) - g(x,u(0)(x),0)], \qquad x \in \Omega\,,\\ &v(x) = 0\,, \quad x \in \partial\Omega\,. \end{aligned} \tag{6}$$

where $\lambda \in \mathbb{R}$ (think that (5) is obtained from (6) for $\lambda = 0$). It may be proved that we can transform (6) to apply a global bifurcation theorem by Rabinowitz [9]. Conditions (3) guarantee that, for $\lambda = 0$, (6) has a nontrivial and nonnegative solution v such that $u(v) \not\equiv 0$.

Remarks and an example. There is a vast literature about the existence of positive solutions of problems like (2). Our Theorem covers, such as it is shown in [1], many previous results obtained independently for the three standard models (competition, prey–predator and cooperative ones) and some others as, for instance, that of example bellow. The main novelties of our result are two: first we do not suppose any kind of monotonicity of f respect to v and of g respect to u, and second, we allow f and g to have spatial dependence.

Example. Consider the problem

$$(7) \qquad \begin{aligned} -\Delta u(x) &= u(x)\,(a - bu(x) - cv(x))\,, \quad x \in \Omega, \\ -\Delta v(x) &= v(x)\,(e - fv(x) + g(x)u(x))\,, \quad x \in \Omega, \\ u(x) &= v(x) = 0\,, \quad x \in \partial\Omega, \end{aligned}$$

where $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, b,c,f are positive constants, and $g : \overline{\Omega} \longrightarrow \mathbb{R}$ is defined by $g(x_1,x_2) = x_1$, $\forall (x_1,x_2) \in \overline{\Omega}$. Then, if $e \geqslant \lambda_1(\Omega,0)$ and $a > \lambda_1(\Omega,cv(0))$, (7) has a coexistence state. (For instance, this is the case if $e \geqslant \lambda_1(\Omega,0)$ and a is sufficiently large).

Observe that (7) is of competition type in Ω_1 and of prey–predator type in Ω_2, where

$$\Omega_1 = \{x \in \Omega : x_1 < 0\} \qquad \text{and} \qquad \Omega_2 = \{x \in \Omega : x_1 > 0\}.$$

Finally we would like to bring up a question which we consider quite interesting: let us suppose that f, g are defined on $\mathbb{R}^n \times [0,+\infty) \times [0,+\infty)$, then, how do the conditions (3) change if Ω changes?. A satisfactory answer to this question would allow to obtain domains of coexistence for (2) starting from a concrete domain of coexistence. Some partial answers to this last question have been given in [2], [3], where the case of $\Omega = B(0;R)$ (the ball in $\mathbb{R}^n$ of center zero and radious $R > 0$) has been specially considered.

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