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Dowker–Type Theorems in Finite Dimensional Spaces

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INTRODUCTION

Let C be a subset of a metric space. We shall say that C is a convex body if it is a compact convex set with a non–empty interior.

In $\mathbb{R}^m$ with the norm induced by the usual scalar ptoduct, there are some estimators, not necessarily a metric, of "the distance" between convex bodies. In what follows d shall be one of those estimator functions.

For $n = m+1, m+2, \ldots$ let $\mathcal{P}_n$ be the set of all convex polytopes having at most n vertices. Given a convex body C we define $\mathcal{P}_n{}^i(C)$ to be the set of poly–topes of $\mathcal{P}_n$ contained in C and let similarly $\mathcal{P}_n{}^c(C)$ denote the set of polytopes of $\mathcal{P}_n$ containing C. We shall write $\mathcal{P}_n{}^i$, $\mathcal{P}_n{}^c$ instead of $\mathcal{P}_n{}^i(C), \mathcal{P}_n{}^c(C)$.

Let $\delta : \mathbb{N} \longrightarrow \mathbb{R}$ the function defined by $\delta(n) = \inf\{d(C,P) : P \in \mathcal{P}_n(C)\}$, where C is a fixed convex body. In the same way shall be defined the functions $\delta^c(n)$ and $\delta^i(n)$, when $P \in \mathcal{P}_n{}^c(C)$, $\mathcal{P}_n{}^i(C)$ respectively. When referring to the three functions at one time, they will be called $\delta(n)$.

Given C and P, two arbitrary convex bodies, let us define the functions:

$$\rho_1(C,P) = \sup_{x \in C} \inf_{y \in P} |x-y| \qquad \text{and} \qquad \rho_2(C,P) = \rho_1(P,C)\,.$$

If it is clear which are the convex bodies we refer to, we shall denote these functions as ρ_1 and ρ_2

Some classical theorems of Dowker [1] about packing and covering problems promoted the study of the convexity of this type of functions.

Eggleston in [2], while working on those topics, constructed a convex body C in $\mathbb{R}^2$ for which the $\delta(n)$ functions, when $d(C,P) = \delta^E(C,P) = \rho_1 + \rho_2$, are not convex. On the other hand Gruber [3] leaves it open the question of whether or not the $\delta(n)$ functions are convex, when $d(C,P) = \delta^H(C,P) = \max\{\rho_1, \rho_2\}$, the Hausdorff distance.

In this paper we answer this question negatively. We shall prove that for the Eggleston's polygon the $\delta(n)$ functions are not convex. We will also construct in $\mathbb{R}^m$, $m \geqslant 2$, a convex body for which the $\delta(n)$ functions are not convex, when $d(C,P) = f(\rho_1, \rho_2)$, where f is a function belonging to $\Omega = \{f: \mathbb{R}^+ \times \mathbb{R}^+ \cup \{(0,0) \longrightarrow \mathbb{R} : f(x,0) = x, f(0,y) = y, \max\{x,y\} \leqslant f(x,y)\}$.

The functions $f_\lambda(x,y) = (x^\lambda + y^\lambda)^{1/\lambda}$, $\lambda \in (0,\infty)$ and $f_\infty(x,y) = \max\{x,y\}$ belong to Ω.

RESULTS

The case $\mathbb{R}^2$

The convex body C fixed in Eggleston's example, is a regular $2r$ sided polygon of side-length k, and let X_{2r} denote it. We shall see that for X_{2r} the $\delta(n)$ functions are not convex when $f \in \Omega$.

LEMMA 1. *There exists $W \in \mathcal{P}_r^{\,i}$ and $Z \in \mathcal{P}_r^{\,c}$ such that*

$$\rho_1(X_{2r}, W) = K \sin(\pi/2r)\,, \tag{1}$$

$$\rho_2(X_{2r}, Z) = \frac{K}{2} \tan(\pi/r)\,. \tag{2}$$

Proof. To prove (1) let W be the polygon formed by joining r alternate vertices of X_{2r}. For (2) we take Z, a polygon formed by producing r alternate sides of X_{2r}. ∎

Now we take r large enough to let us construct two polygons in which the following lemma is based on.

LEMMA 2. *For r large enough, the following is true*:

$$\delta(2r-1) \geqslant K \sin(\pi/r)/\{4[1 + \cos^2(\pi/2r)]\} = \beta\,. \tag{3}$$

Proof. We begin by constructing two regular polygons of $2r$ sides, X' and X'' such that $X' \subset X_{2r} \subset X''$. If l_i', l_i, l_i'' are the sides of X', X_{2r}, X'' respectively, then the polygons will verify that, for each i, l_i', l_i, l_i'' are parallel and β denotes the distance between l_i' and l_i, and between l_i'' and l_i.

Take a vertex v_i'' of X'' formed by the sides l_i'' and l_{i+1}'', and construct the triangle C_i joining v_i'' to the mid-points of l_i'' and l_{i+1}'', which shall be denoted by h_i and h_{i+1} respectively. These triangles have disjoint interiors.

The value of β has been chosen so that the polygon formed by joining h_i on adjacent sides is also the polygon formed by the bisectors of the angles $\angle(l'_{i+1}, l'_i)$.

Let L be a polygon such that $d(X_{2r}, L) < \beta$ and let v_i be the vertex of X_{2r} contained in the interior of C_i. The open ball $B(v_i, \beta) \subset \text{int}(C_i)$. From the definition of d, there exists $f \in \Omega$ such that $d(X_{2r}, L) = f(\rho_1(X_{2r}, L), \rho_2(X_{2r}, L))$ therefore $f(\rho_1(X_{2r}, L), \rho_2(X_{2r}, L)) < \beta$ from where

(a) $\rho_1(X_{2r}, L) < \beta \implies$ there exists $z \in L$ such that $z \in B(v_i, \beta) \subset \text{int}(C_i)$,

(b) $\rho_2(X_{2r}, L) < \beta$, therefore $L \subset X''$.

From (a) we conclude that the straight line joining h_i and h_{i+1} meets in $\text{int}(L)$, and from (b) L has at least one vertex in each $\text{int}(C_i)$. Therefore L has at least $2r$ vertices. This proves boundary (3). ∎

To finish the proof, one must bear in mind the inequalities $\delta(n) \leqslant \delta^i(n)$ and $\delta(n) \leqslant \delta^c(n)$, and suppose that $\delta(n)$ is a convex function. Then using the lemmas 1, 2 and considering that a convex function $g(n)$ of the integral variable n, such that $g(p) = 0$, verifies that $g(p-r) \geqslant r g(p-1)$ $(r = 1, \ldots, p-1)$, the following inequalities are true

$$K \sin(\pi/2r) = \rho_1(X_{2r}, W) \geqslant \delta^i(r) \geqslant \delta(r) \geqslant r\delta(2r-1) \geqslant$$
$$\geqslant r K \sin(\pi/r)/\{4[1 + \cos^2(\pi/2r)]\}\,. \qquad (4)$$

But inequalitie (4) is false if r is large. Similarly, it may be shown that $\delta^i(n)$ and $\delta^c(n)$ are not convex.

The case $\mathbb{R}^3$

We are going to construct in $\mathbb{R}^3$ a polytope with $4r$ vertices for which the $\delta(n)$ functions, when $d(C, P) = f(\rho_1, \rho_2)$, and $f \in \Omega$, are not convex. This polytope shall be denoted by X_{4r} and will be constructed based on X_{2r}. Through induction on m, a convex body in $\mathbb{R}^m$, for which the $\delta(n)$ functions, when $d(C, P) = f(\rho_1, \rho_2)$ and $f \in \Omega$, are not convex.

DEFINITION 1. We say that $A \subset \mathbb{R}^n$ is a *straight polytope* with $n-1$ dimensional bases and height ξ, if there exists a hyperplane H in $\mathbb{R}^n$ which contains a polytope B such that $A = \{x + \lambda u : x \in B \text{ and } |\lambda| \leqslant \xi\}$, where u is a vector of norm 1, orthogonal to H.

We consider in $\mathbb{R}^3$ the $z = 0$ plane and X_{2r} contained in it. We are going to work with the poiytope $X_{4r} = \{x + \lambda u : x \in X_{2r} \text{ and } |\lambda| \leqslant \xi\}$, when $u = (0, 0, 1)$.

LEMMA 3. *There exists* $W^* \in \mathcal{P}_{2r}^{i}$ *and* $Z^* \in \mathcal{P}_{2r}^{c}$ *such that*

$$\rho_1(X_{4r}, W^*) = K\sin(\pi/2r)\,, \tag{5}$$

$$\rho_2(X_{4r}, Z^*) = \frac{K}{2}\tan(\pi/r)\,. \tag{6}$$

Proof. To prove (5), construct a straight polytope W^* with base W (see (1)) and height ξ. To demostrate (6), construct a different straight polytope Z^* with base Z (see (2)) and height ξ. ∎

In the proof of the following inequality a family of convex bodies appears. From a certain large value of ξ, it is possible to affirm that each convex body contains a ball with centre at a vertex of X_{4r} and radius β, so that the interiors of the convex bodies are disjoint.

LEMMA 4.

$$\delta(4r-1) \geqslant K\sin(\pi/r)/\{4[1+\cos^2(\pi/2r)]\} = \beta\,. \tag{7}$$

Proof. Construct the straight polytopes $X' = \{x+\lambda u : x \in X_r'$ and $|\lambda| \leqslant \xi-\beta\}$ and $X'' = \{x+\lambda u : x \in X_r''$ and $|\lambda| \leqslant \xi+\beta\}$, where X_r' and X_r'' are the X' and X'' constructed in the proof of Lemma 3.

We shall construct a family of $4r$ convex bodies with disjoint interiors, and such that each open ball with centre at a vertex of X_{4r} and radius β is contained inside one of the convex bodies. Therefore, reasoning as in the proof of Lemma 2, any polytope L such that $d(X_{4r}, L) < \beta$ would have at least $4r$ vertices. This prove (7).

The convex bodies are defined in the following way: let C_i be the tetrahedron whose vertices are v_i'', a vertex of X'', and the mid-points on v_i'' edges. We can assert that $B(v_i'', \beta) \subset \operatorname{int}(C_i)$ if ξ is large, and that the interiors of C_i are disjoint. ∎

For $n = 4r$, $\delta(n) = 0$, if $\delta(n)$ was convex, reasoning as we did at the end of the case $\mathbb{R}^2$, one would obtain the iniquality

$$K\sin(\pi/2r) \geqslant 2r\,K\sin(\pi/r)/\{4[1+\cos^2(\pi/2r)]\}$$

which is false for large r. Then $\delta(n)$ cannot be convex. Analogously, it can be shown that $\delta^{i}(n)$ and $\delta^{c}(n)$ are not convex.

Remarks. An example in $\mathbb{R}^m$ can be constructed by induction.

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