EXTRACTA MATHEMATICAE Vol. 7, Núm. 2-3, 119 – 125 (1992)

A Correspondence between Finite–dimensional Lie Superalgebras and Supergroups

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AMS *Subject Class.* (1991): 17B70, 22E65, 58A50. *Received September* 29, 1992

INTRODUCTION

Supergoups are in the very heart of both pure and applied supermathematics. Those objects have been introduced "to recapture explicitely the geometry implicit in the algebraic structure of Lie superalgebras" [5], and their significance for theoretical physics is illuminated, for example, in [13, 25, 32] etc.

Nevertheless, the "soft" super Lie theory so far lacks a comprehensive superanalog of Lie–Cartan theorem. To realize the nature of the difficulties, it should be remembered that one may consider Lie superalgebras not only over basic fields such as $\mathbb{R}$ or $\mathbb{C}$ but also over arbitrary graded commutative algebras Λ. The corresponding concept of supermanifolds and supergoups over nontrivial ground algebras Λ was best presented by Rothstein [33, 34], see also [2 – 4, 11, 12, 13, 15, 16, 24, 35, 36, 41]; it is believed by many superanalysts that this approach can be reformulated in the language of relative categories over prime spectra *Spec* Λ (cf. [26]), but there is still a long way to go. To construct supergroups over an algebra Λ other than $\mathbb{R}$ or $\mathbb{C}$, one may apply the base change functor [9] to supergroups over *Spec* $\mathbb{C}$ (or *Spec* $\mathbb{R}$). (Although even here one should surmount certain difficulties of functional–analytic nature in the case where Λ is infinite–dimensional.) The resulting supergroups correspond to Lie superalgebras of the type $\mathfrak{g} \simeq \Lambda \oplus \mathfrak{g}'$ where $\mathfrak{g}'$ is an arbitrary Lie superalgebras over $\mathbb{K}$. Berezin [7] referred to such Lie superalgebras as "Grassmann envelopes" of Lie superalgebras over $\mathbb{K}$, and DeWitt [13] as "conventional" Lie superalgebras. However, not all Lie superalgebra $\mathfrak{g}$ over graded–commutative algebras Λ are of such type, and in general a Lie superalgebra $\mathfrak{g}$ over Λ can be viewed only as an appropriate

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deformation over the base *Spec* Λ of a Lie $\mathbb{K}$–superalgebra [14]. Thus, the problem of associating a supergroup to a Lie superalgebra $\mathfrak{g}$ over an algebra Λ remains still actual and in no way it can be reduced to the similar problem in the absolute case $\Lambda = \mathbb{R}$ or $\mathbb{C}$ treated by Berezin and Leĭtes [8] and Kostant [19]. The problem has already been solved in the positive in the cases where Λ is a finite–dimensional Grassmann algebra [36,4], and the infinite dimensional Banach–Grassmann algebra B_∞ [10,27].

Here we claim the validity of such a result for Lie superalgebras over ground algebras Λ of a fairly general nature. In particular, a number of examples of graded–commutative algebras used in superanalysis [3,6,10–13,15,16,24,35] is included. Our results can be viewed as a step towards solving the DeWitt problem [13] of classifying all supergroups corresponding to a given Lie superalgebras. We basically follow the Rothstein approach to supermanifolds [33,34], although the class of ground algebras under consideration is wider than there; our exposition somewhat differs from [33]. A graded commutative associative unital algebra Λ is called a *graded local algebra* [11] if it possesses a unique maximal ideal J_Λ. In this case $\Lambda / J_\Lambda \simeq \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). If a graded local algebra Λ is Hausdorff topological then J_Λ is necessarily closed and Λ decomposes into a direct sum $\Lambda \simeq \mathbb{K} \oplus J_\Lambda$; the corresponding *body* and *soul* projections are denoted by $\beta_\Lambda : \Lambda \longrightarrow \mathbb{K}$ and $\sigma_\Lambda : \Lambda \longrightarrow \mathbb{K}$ respectively. An *Arens–Michael algebra* [18] is a complete *locally multiplicatively convex* algebra [1,22]; the local multiplicative convexity of a topological algebra Λ means that it can be embedded into a Tychonoff product of a family of Banach algebras as a topological subalgebra. We restrict the class of ground algebras Λ of "superconstants" to the *graded local Arens–Michael* (for short, *GLAM*) algebras; there is a strong evidence that GLAM algebras form an adequate ground for dealing with superanalyticity in finite dimensions. All the ground algebras so far used in superanalysis are GLAM algebras. Among the main examples are: Banach–Grassmann algebras [16], including the algebra B_∞ [35]; Grassmann– Banach algebras [15], including finite–dimensional Grassmann algebras $\bigwedge(q)$; the complete metrizable DeWitt algebra Λ_∞ [13,12]; the nuclear *(LB)*–algebra $\bigwedge(\infty)$ [24]. The "Grassmann algebras with infinitely many generators" introduced by Berezin [6; Ch. 1, Sec. 3.3] are GLAM algebras as soon as they are locally convex. GLAM algebras also come into being as algebras of superanalytic super functions over purely odd graded locally convex space, see [37]. For more on this, including a number of new examples and constructions of GLAM algebras, see [29,30].

Throughout the present note, Λ stands for an arbitrary GLAM algebra.

By a *Rothstein superspace* over Λ we call a triple $M = (M_0, \mathcal{O}_M, ev_M)$ where $\mathcal{O}_M$ is a sheaf of graded Λ algebras on a topological space M_0 and $ev_M : \mathcal{O}_M \longrightarrow \mathcal{C}_M$ is a sheaf morphism (here $\mathcal{C}_M$ denotes the sheaf of germs of all continuous Λ valued functions on M_0). A *morphism* $\phi : M \longrightarrow N$ between Rothstein superspace is a pair $(\phi_0, \phi\#)$ where $\phi_0 : M_0 \longrightarrow N_0$ is a continuous mapping, $\phi\# : \mathcal{O}_N \longrightarrow \phi_{0*}\mathcal{O}_M$ is a sheaf morphism, and the four sheaf morphisms ϕ_0^*, $\phi\#$, ev_M, ev_N agree with each other in a clear way. (Cf. [28,2,3].) Morphisms of a one–point superspace Υ (over Λ) to M are referred to as Υ *points* of M and their totality is denoted by $pt_\Upsilon M$.

Remarks. Any Rothstein superspace determines an object from the relative category **Ss** $\downarrow$ *Spec* Λ where **Ss** stands for the category of locally ringed superspaces [21], cf. [26]. In particular, Rothstein superspace over the basic field $\mathbb{K}$ are exactly superspaces from the category **Ss** $\downarrow \mathbb{K}$ [26]. However, in general case a Rothstein superspace bears some additional structure (the evaluation morphism) needed to make the notion of a supermanifold consistent (cf. examples in [33,34]). It is our belief that the most important open problem of superanalysis is to establish an exact connection between the present approach and the relative one.

Any finite–dimensional graded linear space $V = V^0 \oplus V^1$ determines a Rothstein superspace over any Λ which we denote by $V(\Lambda)$. Namely, $V(\Lambda)_0 := (\Lambda \otimes V)^0$, the sheaf $\mathcal{O}_{V(\Lambda)}$ is a constant sheaf of graded symmetric Λ algebras over the graded Λ module $\Lambda \otimes V'$ (for basic notions of graded algebra, consult any of the treatises [3,20,21,26]), and the evaluation morphism $ev_{V(\Lambda)}$ is determined by letting $ev_{V(\Lambda)} \lambda \otimes f(\mu \otimes x) := (-1)^{\tilde{f}\tilde{\mu}} \lambda \mu f(x)$. We call $V(\Lambda)$ the *affine superspace* associated with $\mathbb{K}^{0|n}$ (cf. [34]). A set $U \subset V(\Lambda)_0$ is said to be *DeWitt open* if it is open in the usual projective tensor product topology and in addition coincides with $\beta_{V(\Lambda)}^{-1} \beta_{V(\Lambda)}(U)$ where $\beta_{V(\Lambda)} = \beta_\Lambda \otimes id_V$.

For any $U \subset V(\Lambda)_0$ open, the topology of *supersimple convergence* on the algebra of superpolynomials $\mathcal{O}_{V(\Lambda)}(U)$ is determined by letting a net (f_α) of superpolynomials indexed with elements of a directed partially ordered set [17] to convergence to f if and only if for each $n \in \mathbb{N}$ and each $x \in pt_n U$ the following holds: $x\#(f_\alpha) \longrightarrow x\#(f)$. Using this topology and the notion of the order of a superpolynomial at a point, one can define the sheah $\mathcal{A}_U$ of germs of superanalytic super-functions on U and to extend to it the morphism ev_U in the most direct way [28].

The resulting Rothstein superspace $(U, \mathcal{A}_U, ev_U)$ is called a *superanalytic superdomain* (of dimension *dim V*). Any Rothstein superspace X locally isomorphic to a superdomain is called a *superanalytic supermanifold*. If all the model superdomains U may be chosen so that their underlying sets U_0 are DeWitt open then X is called a (superanalytic) *DeWitt supermanifold* (cf. [3,31]).

A *(DeWitt) supergroup* is a group object in the category of (DeWitt) superanalytic supermanifolds and Rothstein superspace morphisms.

By a (finite–dimensional) *Lie superalgebra* over Λ one means a free graded Λ module (that is, that of the form $\Lambda \otimes V$) endowed with a bi Λ linear graded antisymmetric "super Lie bracket" satisfying the graded Jacobi identity [3,13, 21,26]. The *Lie superalgebra* $sLie(G)$ *of a supergroup* G is formed by all graded vector fields X on G (that is, graded derivations of the structure sheaf $\mathcal{A}_G$) which are left–invariant in the sense that $(Id \otimes X)\mu\# f = \mu\#(Xf)$ where μ is the multiplication morphism (cf. [8,9,3,26]).

THEOREM 1. *For any Lie superalgebra* $\mathfrak{g}$ *over a GLAM algebra* Λ *the Schur–Baker–Campbell–Hausdorff–Dynkin (SBCHD) series converges on a DeWitt open neighbourhood* U *of the origin of the even part* $\mathfrak{g}^0$ *making* U *into a local analytic Lie group assigned to a locally convex Lie algebra* $\mathfrak{g}^0$.

THEOREM 2. *The even part* $\mathfrak{g}^0$ *of a Lie superalgebra* $\mathfrak{g}$ *over a GLAM algebra* Λ *is enlargible as a locally convex Lie algebra. Moreover, one can enlarge* $\mathfrak{g}^0$ *to a Baker–Campbell–Hausdorff Lie group* G *in the sense of Milnor* [23] *in such a way that the restriction of* exp_G *to a DeWitt open neighbourhood of the origin in* $\mathfrak{g}^0$ *is an injection.*

Proof (sketch). The SBCHD series makes the Lie ideal $J^0_{\mathfrak{g}} := \beta^{-1}_{\mathfrak{g}^0}(0)$ into an analytic Lie group which we denote by $J^+_{\mathfrak{g}}$. Every operator adjoint to an element of $J^+_{\mathfrak{g}}$ is quasinilpotent, so it is an exponential Lie group. The techniques developed by van Est [40] and Świerczkowski [39] allows one to fill the gap in the sequence

$$e \longrightarrow J^+_{\mathfrak{g}} \longrightarrow \ ? \longrightarrow H \longrightarrow e$$

were H is a simply connected Lie group corresponding to the finite–dimensional Lie algebra $\mathfrak{g}^0_B := \mathfrak{g}^0 / J^0_{\mathfrak{g}}$, in a way agreeable to the original Lie algebra sequence

$$0 \longrightarrow J_{\mathfrak{g}}^{0} \longrightarrow \mathfrak{g}^{0} \longrightarrow \mathfrak{g}^{0}/J_{\mathfrak{g}}^{0} \longrightarrow 0 .$$

The corresponding exponential mapping turns out to be injective on a DeWitt open neighbourhood of the origin in $\mathfrak{g}^0$ of the form $V \times J_{\mathfrak{g}}^{+}$ where V is a domain of injectivity of exp_H. Finally, Theorem 1 together with the Theorem on extension of analytic structure [38] is applied. At all stage it is more convenient to work not with Lie algebras themselves but rather with their Banach–Lie quotients, and then to apply the projective spectra techniques, using the fact that $\mathfrak{g}^0$ is an Arens–Michael Lie algebra. ∎

MAIN THEOREM. *An arbitrary finite–dimensional Lie superalgebra* $\mathfrak{g}$ *over a GLAM algebra* Λ *comes from an essentially unique DeWitt superanalytic supergroup* G *with a connected simply connected underlying topological group* G_0.

Proof (sketch). Applying Theorem 2 to Lie superalgebras $\mathfrak{g}_k := \mathfrak{g} \otimes_{\mathbb{K}} \Lambda(k)$, one obtains a collection on Baker–Campbell–Hausdorff Lie groups G_k and analytic mappings $exp_k : \mathfrak{g}_k^0 \longrightarrow G_k$, $k \in \mathbb{N}$. Those allow one to transfer from $\mathfrak{g}^0$ to G_0 the structure of a superanalytic supermanifold and to define structure morphisms in such a manner that $G_k = pt_k G$ where G is a supergroup resulting from the process. It turns out that $sLie(G) \simeq \mathfrak{g}$. The uniqueness of G follows from the uniqueness of Lie groups G_k (or, rather, their Banach–Lie quotients). ∎

Final Remarks. Proofs of Theorems 1 and 2 are to found in [30]. Proof of the Main Theorem in some aspects follows the proof of the same result in the case $\Lambda = \mathbb{K}$ contained in [9]; however, more subtle techniques developed by Rothstein [34] are involved.

ACKNOWLEDGEMENT

This investigation was at its early stages financially supported by the Tomsk State University Doctorate Program, and basically completed during visits of the author to the Univerity of Genoa in November 1990 and May–July 1991. The author is most grateful to Professor Ugo Bruzzo for organizing those visits and for his utmost hospitality at the Department of Mathematics of the University of Genoa. The visits were financially supported by the Mathematical Physics Group of the Italian National Research Council through the Visiting Professorship Scheme. The author also acknowledges certain electrifying comments from Professor D. Leites.

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