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The Space of Compact Operators as an M–Ideal in its Bidual

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INTRODUCTION.

A Banach space X is said to be an M–ideal in its bidual if the canonical decomposition $X^{***}=X^{*}\oplus X^{\perp}$ is an ℓ^1–direct sum. These spaces enjoy some remarkable topological properties. For example, for any such X, X^* has the Radon Nikodym property [9] and X has the Pełczyński property $(\mathcal{U})$ [7] and X is weakly compactly generated [5].

Harmand and Lima [9] have proved that for a reflexive Banach space X, if $\mathcal{K}(X)$ the space of compact operators is an M–ideal in $\mathcal{L}(X)$ the space of bounded operators then $\mathcal{L}(X)$ is indeed the bidual of $\mathcal{K}(X)$ and hence $\mathcal{K}(X)$ is an M–ideal in its bidual. This result has recently been extended in [4] to obtain the same conclusion for $\mathcal{K}(X,Y)$ when X and Y are reflexive Banach spaces and $\mathcal{K}(X,Y)$ is an M–ideal in $\mathcal{L}(X,Y)$.

In this paper we exhibit several classes of Banach spaces for which $\mathcal{K}(X,Y)$ is an M–ideal in its bidual so that $\mathcal{K}(X,Y)$ enjoys the nice topological properties some of which have been mentioned above. See also [14].

We refer the reader to [2] for relevant definitions and results of M–structure theory that we will be using here and the forth coming monograph [10] and its exhaustive bibliography for examples and properties of Banach spaces that are M–ideals in their biduals.

We shall be repeatedly making use of the following theorem where part A) has been proved in [9] and part B) very recently in [12].

THEOREM. *Let X be a Banach space.*

A) *If X is an M–ideal in its bidual then for any closed subspace $Y\subset X$, Y is an M–ideal in its bidual.*

B) *If X is such that every separable Banach subspace of X is an M–ideal in* *its bidual then X is an M–ideal in its bidual.*

MAIN RESULTS.

Since X^* and Y are isometric to subspaces of $\mathscr{K}(X,Y)$, by A) of the above theorem we see that for $\mathscr{K}(X,Y)$ to be an M–ideal in its bidual it is necessary that both X^* and Y be M–ideals in their biduals and appealing to Corollary 3.7 of [9], as was done in [9] we conclude that it is necessary that X is reflexive and Y is an M–ideal in its bidual.

We first look at the situation when X and Y are reflexive and present an argument that gives a simple geometric proof of the main result of [4].

PROPOSITION 1. *Suppose X and Y are reflexive Banach spaces and $\mathscr{K}(X,Y)$ is an M–ideal in $\mathscr{L}(X,Y)$ then $\mathscr{L}(X,Y)$ is the bidual of $\mathscr{K}(X,Y)$.*

Proof. By hypothesis we have

$$\mathscr{L}(X,Y)^* = \mathscr{K}(X,Y)^* \oplus_1 \mathscr{K}(X,Y)^{\perp}.$$

However since functionals in the unit ball of $\mathscr{K}(X,Y)^*$ determine the norm of any operator we conclude that the canonical embedding of $\mathscr{L}(X,Y)$ into $\mathscr{K}(X,Y)^{**}$ is an isometry. That this isometry is onto follows from the results of Feder and Saphar [6]. ∎

THEOREM 1. *Suppose that X and Y are reflexive Banach spaces and $\mathscr{K}(X,Y)$ is an M–ideal in $\mathscr{L}(X,Y)$ and suppose further X has the compact approximation property then for any closed subspace $Z \subset Y$, $\mathscr{K}(X,Z)$ is an M–ideal in $\mathscr{L}(X,Z)$ and dually if Y has the compact approximation property then for any closed subspace $M \subset X$, $\mathscr{K}(X/M,Y)$ is an M–ideal in $\mathscr{L}(X/M,Y)$.*

Proof. Since X and Y are reflexive it follows from the results of [6] that

$$\mathscr{K}(X,Y) \subset \mathscr{K}(X,Y)^{**} \subset \mathscr{L}(X,Y).$$

From the hypothesis we known that $\mathscr{K}(X,Y)$ is an M–ideal in its bidual.

Since $\mathscr{K}(X,Z) \subset \mathscr{K}(X,Y)$ we conclude that $\mathscr{K}(X,Z)$ is an M–ideal in its bidual. Now since X has the compact approximation property, invoking Corollary 1.3 of [8] we get that $\mathscr{K}(X,Z)^{**} = \mathscr{L}(X,Z)$ and hence $\mathscr{K}(X,Z)$ is an M–ideal in $\mathscr{L}(X,Z)$.

To see the dual statement we observe first that since Y is reflexive, Y^* has the compact approximation property and the map $T \longrightarrow T^*$ is an onto isometry from the operator spaces $\mathscr{K}(X/M,Y)$ ($\mathscr{L}(X/M,Y)$) and $\mathscr{K}(Y^*,M^\perp)$ ($\mathscr{L}(Y^*,M^\perp)$) therefore the conclusion follows from the first part of this theorem and this observation. ∎

COROLLARY. *Let X be reflexive and $\mathscr{K}(X)$ an M–ideal in $\mathscr{L}(X)$ then for any $Z\subset X$, $\mathscr{K}(X,Z)$ is an M–ideal in $\mathscr{L}(X,Z)$ and $\mathscr{K}(X|Z,X)$ is an M–ideal in $\mathscr{L}(X|Z,X)$.*

Proof. It follows from Lemma 5.1 of [9] that X has the compact approximation property. ∎

Remark. It should be noted that these conclusion can also be drawn from a more general approach involving properties of compact operator spaces as M–ideals, as was done in Proposition 2.9 of [12].

From now on we assume that Y is a non–reflexive space that is an M–ideal in its bidual and X is a reflexive Banach space. Note that we still have from the results of Feder and Saphar [6]

$$\mathscr{K}(X,Y)^{**}\subset\mathscr{L}(X,Y^{**}).$$

Let us also note here that $\mathscr{L}(X,Y^{**})$ is isometric to $\mathscr{L}(Y^*,X^*)$ by the map $T\longrightarrow T^*|Y^*$ (this is true for any Banach spaces X and Y). ∎

PROPOSITION 2. *Let Y be such that for all Banach space Z, $\mathscr{K}(Z,Y)$ is an M–ideal in $\mathscr{L}(Z,Y)$ then for any reflexive Banach space X, $\mathscr{K}(X,Y)$ is an M–ideal in its bidual.*

Proof. The class of Banach spaces Y described above is the so called M_∞ spaces studied in [13], [10] (Y is non–reflexive when it is infinite dimensional). It follows from the special compact approximation of the identity enjoyed by these spaces (see [10] Chapter 6) that for any such Y, $\mathscr{K}(Z,Y)$ is also an M–ideal in $\mathscr{L}(Z,Y^{**})$.

Hence when X is a reflexive Banach space from the results of Feder and Saphar alluded to before we have

$$\mathscr{K}(X,Y)\subset\mathscr{K}(X,Y)^{**}\subset\mathscr{L}(X,Y^{**})$$

and hence $\mathscr{K}(X,Y)$ is an M–ideal in its bidual. ∎

Remark. It is known that the class of M_∞ spaces is not closed under subspaces, however if $Y\in M_\infty$ and $Z\subset Y$ is a closed subspace then since $\mathscr{K}(X,Z)$

$\subset \mathscr{K}(X,Y)$ we conclude that $\mathscr{K}(X,Z)$ is an M–ideal in its bidual for such a Z and for any reflexive Banach space X.

The authors in [12] study a class of Banach spaces closely related to the M_∞ spaces. These are Banach spaces Y with the property that $\mathscr{K}(\ell^1,Y)$ is an M–ideal in $\mathscr{L}(\ell^1,Y)$. Our final result concerns this class.

THEOREM 2. *Let Y be a Banach space such that Y has the compact metric approximation property and $\mathscr{K}(\ell^1,Y)$ is an M–ideal in $\mathscr{L}(\ell^1,Y)$ then for any reflexive Banach space X, $\mathscr{K}(X,Y)$ is an M–ideal in its bidual.*

Proof. In view of B) of the Theorem quoted above, we only need to show that every separable subspace S of $\mathscr{K}(X,Y)$ is an M–ideal in its bidual. Let $S\subset \mathscr{K}(X,Y)$, S a separable subspace. W.l.o.g. assume that $S\subset \mathscr{K}(X,Z)$ where $Z\subset Y$ and Z is a separable Banach space. Since the space Y is an M–ideal in its bidual ((a) of Theorem 2.12 [12]) it is weakly compactly generated and hence by a result of Amir and Lindenstrauss [1], there is a separable subspace Z' of Y which is 1–complemented in Y such that

$$Z\subset Z'\subset Y.$$

Note that Z' has now the metric compact approximation prperty and $\mathscr{K}(\ell^1,Z')$ is an M–ideal in $\mathscr{L}(\ell^1,Z')$, (see [11]). Therefore by c) Theorem 2.12 [12] we get that Z' is in the class M_∞. Hence by the remark made above we conclude that $\mathscr{K}(X,Z)$ is an M–ideal in its bidual.

There is a natural way of generating more examples of this class we mention without proof that if $\{Y_\alpha\}$ is a family of Banach spaces such that $\mathscr{K}(X,Y_\alpha)$ is an M–ideal in its bidual then $\mathscr{K}(X,\oplus_{c_0}Y_\alpha)$ is an M–ideal in its bidual.

From what we saw above for reflexive spaces with the compact approximation property, the space of compact operators is an M–ideal in the bidual is equivalent to the space of compact operator being an M–ideal in the space of bounded operator. It is well known (see [10]) that for $X=L^p[0,1]$, $p\neq 2$, $K(X)$ is not an M–ideal in $L(X)$ and hence $K(X)$ is not an M–ideal in its bidual. So by taking $Y=X\otimes_\infty c_0$ we get a non–reflexive Banach space that is an M–ideal in its bidual for which $K(X,Y)$ is not an M–ideal in its bidual (I am greatful to Dirk Werner for this remark).

Since the injective tensor product $X\otimes_\varepsilon Y$ of two M_∞–spaces X and Y is again an M_∞–space ([10], Chapter 6), if Y is as in Theorem 2 and X a subspace of an M_∞–space or a reflexive space then arguments similar to the one given during the proof of Theorem 3 yield that $X \otimes_\epsilon Y$ is an M–ideal in its bidual. The following question is open.

If Y is a subspace of a M_∞–space, is $X \otimes_\epsilon Y$ an M–ideal in its bidual for any X that is in M–ideal in its bidual?

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