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Some Theoretical Results About the Distribution of a Doubly Stochastic Poisson Process

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1. INTRODUCTION

Poisson processes are suitable models for a broad variety of counting phenomenon. Nevertheless, there are too many situations in which this model is inadequate because of the excessively deterministic character of its intensity function. Let us think in an event such as the radiactive particles detected by a Geiger counter if it is managed over a surface in a random way, as example, as a Brownian motion. In fact, its intensity, is randomized by an external origin and the Poisson model must be revised.

Situations as the above mentioned one drove to Cox [3] to introduce the doubly stochastic Poisson processes (DSPP) as counting processes $\{N(t), t \geqslant t_0\}$ with intensity $\mu_t(x(t))$, depending upon a vectorial random process $\{x(t), t \geqslant t_0\}$ such that, for almost every sample path of this external process, $\{N(t)\}$ is an homogeneous Poisson process with rate $\mu_t(x(t))$. Therefore, the discrete density function is given by

$$
\begin{aligned}
(1) \qquad & P(N(t)=n) = \mathrm{E}[P(N(t)=n/x(s),\ s \geqslant t_0] = \\
& = \frac{1}{n!}\mathrm{E}\left[\left[\int_{t_0}^{t} \mu_s(x(s))\,\mathrm{d}s\right]^n \exp\left\{-\int_{t_0}^{t} \mu_s(x(s))\,\mathrm{d}s\right\}\right].
\end{aligned}
$$

Some important contributions on theoretical and practical aspects of DSPPs are the ones of Asher & Lainiotis [1], Bartlett [2], Grandell [4], Jiménez & Valderrama [5], Konecny [6], Lanska [7], Neuts [8], Ogata & Akaike [9], Rudemo [10] and Snyder & Miller [11], among others.

In this paper we consider that the intensity function follows a Gaussian distribution with positive mean, as an application of the classical central limit theorem to several forces acting on the intensity function as i.i.d. variables. We denote it by $I(t) = \mu_t(x(t))$, being its mean and variance functions $M(t)$ and

$V(t)$ respectively, and the covariance function $R(t,s)$.

2. The Characteristic Function

By operating with the intensity and having into account the symmetry of the covariance function we have

(2) $$\mathrm{E}\left[\left[\int_{t_0}^{t}\{I(s)-M(s)\}\mathrm{d}s\right]^2\right]=\int_{t_0}^{t}\int_{t_0}^{t}R(v,s)\mathrm{d}v\mathrm{d}s=2\int_{t_0}^{t}\int_{t_0}^{s}R(v,s)\mathrm{d}v\mathrm{d}s$$

On the other hand, by the Karhunen–Loève theorem (Wong & Hajek [12]) we can write

(3) $$I(t)=\sum_{k=1}^{\infty}\phi_k(t)\,b_k$$

where $\{\phi_k(t)\}$ is the L^2–orthonormal basis of eigenfunctions of a Fredholm integral equation whose kernel is the covariance function:

$$\lambda\phi(t)=\int_{t_0}^{T}R(t,s)\phi(s)\,\mathrm{d}s,\quad t_0\leqslant t\leqslant T$$

and $\{b_k\}$ is a sequence of independent random variables. By denoting now

$$\mathrm{E}[b_k]=m_k,\ \mathrm{Var}[b_k]=v_k,$$

we have

(4) $$\mathrm{E}\left[\left[\int_{t_0}^{t}\{I(s)-M(s)\}\,\mathrm{d}s\right]^2\right]=\sum_{k=1}^{\infty}\left[\int_{t_0}^{t}\phi_k(s)\,\mathrm{d}s\right]^2 v_k$$

and then, from (2) and (4):

(5) $$\sum_{k=1}^{\infty}\left[\int_{t_0}^{t}\phi_k(s)\,\mathrm{d}s\right]^2 v_k=2\int_{t_0}^{t}\int_{t_0}^{s}R(v,s)\,\mathrm{d}v\mathrm{d}s.$$

We can enunciate the following result:

THEOREM 1. *The characteristic function of a* DSPP $\{N(t),\ t\geqslant t_0\}$ *with Gaussian random intensity* $\{I(t)\}$ *is given by*

(6) $$\Phi_{N(t)}(u)=\exp\left\{(e^{iu}-1)^2\int_{t_0}^{t}\int_{t_0}^{s}R(v,s)\,\mathrm{d}v\mathrm{d}s+(e^{iu}-1)\int_{t_0}^{t}M(s)\mathrm{d}s\right\}.$$

Proof. By using (3) into the general expression of the characteristic function, we can write

$$\Phi_{N(t)}(u)=\mathrm{E}\left[\exp\left\{(e^{iu}-1)\int_{t_0}^{t}I(s)\,\mathrm{d}s\right\}\right]=\prod_{k=1}^{\infty}\Theta_k(u)$$

where

$$\Theta_k(u)=\mathrm{E}\left[\exp\left\{(e^{iu}-1)\left[\int_{t_0}^{t}\phi_k(s)\mathrm{d}s\right]b_k\right\}\right].$$

Simple operations in this expresion on the basis that $b_k \rightsquigarrow N(m_k,v_k)$ give

$$\Phi_{N(t)}(u)=\exp\left\{\frac{1}{2}(e^{iu}-1)^2\sum_{k=1}^{\infty}\left[\int_{t_0}^{t}\phi_k(s)\,\mathrm{d}s\right]^2 v_k+(e^{iu}-1)\sum_{k=1}^{\infty}\left[\int_{t_0}^{t}\phi_k(s)\,\mathrm{d}s\right]m_k\right\}$$

If we apply now the identity (5) and observe that

$$M(t)=\sum_{k=1}^{\infty}\phi_k(t)\,m_k$$

the proof is complete. ∎

From expression (6) of the characteristic function, statistical moments for the DSPP can be calculated as follows

$$\mathrm{E}[N(t)]=\int_{t_0}^{t}M(s)\,\mathrm{d}s,\quad \mathrm{Var}[N(t)]=2\int_{t_0}^{t}\int_{t_0}^{s}R(v,s)\,\mathrm{d}v\mathrm{d}s+\int_{t_0}^{t}M(s)\,\mathrm{d}s.$$

3. Evaluating the Sample–Function Density of a DSPP

The sample–function density of a DSPP $\{N(t),\ t\geqslant t_0\}$ with occurrence times $w_1,\ldots,w_n$ is given by (Snider & Miller [11]):

$$(7)\qquad p(\{N(s),\ t_0\leqslant s\leqslant t\})=\mathrm{E}\left[\exp\left\{-\int_{t_0}^{t}I(s)\,\mathrm{d}s+\int_{t_0}^{t}\log I(s)\,\mathrm{d}N(s)\right\}\right].$$

If we suppose again that $\{I(t)\}$ is a Gaussian process with positive mean value and apply the Karhunen–Loève expansion (3), expression (7) can be simplified as follows:

THEOREM 2. *Under above assumptions the sample–function density for the DSPP is*

$$(8)\qquad p(\{N(s),\ t_0\leqslant s\leqslant t\})=\sum_{\substack{j_1,\ldots,j_n=1\\ k_1,\ldots,k_n=1}}^{\infty}h(j_1,\ldots,j_n,k_1,\ldots,k_n)\prod_{l=1}^{n}\phi_{jl}(w_l)\,\phi_{kl}(w_l),$$

being

$$h(j_1,\ldots,j_n,k_1,\ldots,k_n)=\mathrm{E}\left[b_{j_1}\cdots b_{j_n}\cdot b_{k_1}\cdots b_{k_n}\exp\{-\textstyle\sum_k A_k(t)b_k\}\right].$$

Furthermore, we can write

$$h(j_1,\ldots,j_n,k_1,\ldots,k_n)=\prod_{m=1}^{\infty}\prod_{k=1}^{\infty}F_{b_k}^{(2p_m)}(-A_k(t))$$

where

$$A_k(t)=\int_{t_0}^{t}\phi_k(s)\,\mathrm{d}s$$

and $F_{b_k}(\cdot)$ *is the moment generating function of the Gaussian variable* b_k *so that*

$$F_{b_k}^{(2p_m)}(u) = e^{u^2/2} \sum_{r=0}^{p_m} \frac{V_{2p_m}^{2r}}{(2r)!!} u^{2(p_m-r)}$$

and p_m *denotes the number of subscripts verifying* $j_i = k_i = m$, *so that* $\Sigma_{m=1}^{\infty} p_m = n$.

Proof. The sample–function density (7) can be expressed now as follows

$$p(\{N(s)),\ 0 \leqslant s \leqslant t\}) = \mathrm{E}\left[I(w_1)\, I(w_2) \cdots I(w_n) \exp\left\{-\sum_k A_k(t) b_k\right\}\right],$$

and this expression turns into (8) taking into account that

$$h(j_1,\ldots,j_n,k_1,\ldots,k_n) = \prod_{m=1}^{\infty} \mathrm{E}\left[b_m^{p_m+q_m} \exp\left\{-\sum_k A_k(t) b_k\right\}\right]$$

where p_m and q_m denote the number of subscripts j_i and k_i respectively equal to m, being $\Sigma_m p_m = \Sigma_m q_m = n$. By applying now the moment factorization property of Gaussian random variables we have $h(j,k) = 0$ if $p_m \neq q_m$ and

$$\mathrm{E}\left[(b_m^{p_m})^2 \exp\left\{-\sum_k A_k(t) b_k\right\}\right] = \prod_k \mathrm{E}\left[b_m^{2p_m} \exp\left\{-A_k(t) b_k\right\}\right] = \prod_k F_{b_k}^{(2p_m)}(-A_k(t))$$

and the proof is complete. ∎

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