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Quasi–Metrizable Spaces with a Bicomplete Structure

SALVADOR ROMAGUERA[1] AND SERGIO SALBANY

Escuela de Caminos, Universidad Politécnica, 46071 Valencia, Spain

Department of Mathematics, University of Zimbabwe, Harare, Zimbabwe

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In [1] E. Čech proved that a metrizable space is completely metrizable if and only if it is a G_δ–subset of a compact Hausdorff space in which it is embedded. This result suggests the following concept (see [3, page 252]): A Tychonoff space is called Čech complete if it is a G_δ–subset of each of its compactifications. Then, Čech's theorem can be reformulated as follows [3, Theorem 4.3.26]: A metrizable space is completely metrizable if and only if it is a Čech complete space.

On the other hand, J. Dieudonné showed in [2] that every (pseudo) metrizable space has a compatible complete uniformity.

In this note we solve the problem of extending these classical results to quasi–metrizable spaces.

Terms and concepts which are not defined may be found in [4].

A quasi–pseudometric on a set X is nonnegative real–valued function d on $X \times X$ such that for all $x,y,z \in X$: (i) $d(x,x)=0$ and (ii) $d(x,y) \leqslant d(x,z)+d(z,y)$. If d satisfies the additional condition (iii) $d(x,y)=0 \Leftrightarrow x=y$, then d is called a quasi–metric on X. For each quasi–(pseudo)metric d on X, the function d^{-1}, defined on $X \times X$ by $d^{-1}(x,y)=d(y,x)$, is also a quasi–(pseudo)metric on X called conjugate of d. The quasi–(pseudo)metric d also generates a (pseudo)–metric d^* on X defined by $d^*(x,y)=\max\{d(x,y),d^{-1}(x,y)\}$.

The topology $T(d)$ induced on X by a quasi–pseudometric d on X has as a base the family of d–balls $\{B_d(x,r): x \in X,\ r>0\}$ where $B_d(x,r)=\{y \in X: d(x,y) < r\}$. A topological space (X,T) is called quasi–(pseudo)metrizable if it has a compatible quasi–(pseudo)metric d, where d is called compatible if $T=T(d)$. The notion of a bitopological space appears in a natural way when one considers the topologies $T(d)$ and $T(d^{-1})$ generated on a set X by a quasi–(pseudo)metric

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d and its conjugate. A bitopological space is a triple (X,P,Q) such that X is a nonempty set and P and Q are topologies on X [5]. The space (X,P,Q) is said to be quasi–(pseudo)metrizable if there is a quasi–(pseudo)metric d on X compatible with (P,Q) where d is called compatible if $T(d)=P$ and $T(d^{-1})=Q$.

A quasi–uniform space $(X, \mathcal{U})$ is called bicomplete [4],[6], if the uniformity $\mathcal{U}^*$ is complete (as usual $\mathcal{U}^*$ denotes the coarsest uniformity finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$).

THEOREM 1. *The finest quasi–uniformity of each quasi–pseudometrizable bitopological space is bicomplete.*

Proof. We sketch the proof. Denote by $\mathcal{BFN}$ the finest quasi–uniformity of the quasi–pseudometrizable space (X,P,Q). It suffices to show that $(\mathcal{BFN})^*$ is a complete uniformity. We firstly note that $\mathcal{BFN}$ consists of all $Q\times P$–neighbourhoods of the diagonal. Take a quasi–pseudometric d on X compatible with (P,Q). Let V be a $T(d^*)\times T(d^*)$–neighbourhood of the diagonal. Then, for each $x\in X$ there is a P–neighbourhood P_x of x and a Q–neighbourhood Q_x of x such that $W=\cup\{(Q_x\cap P_x)\times(Q_x\cap P_x):x\in X\}\subseteq V$. For each $x\in X$ select $r_x>0$ such that $B_d(x,4r_x)\subseteq P_x$ and $B_{d^{-1}}(x,4r_x)\subseteq Q_x$. Let $U=\cup\{B_{d^{-1}}(x,r_x)\times B_d(x,r_x):x\in X\}$. Then $V\in(\mathcal{BFN})^*$ since $U\in\mathcal{BFN}$ and $(U\cap U^{-1})\subseteq V$. Consequently $(\mathcal{BFN})^*$ is exactly the fine uniformity of the pseudometrizable space $(X,P\vee Q)$, which is complete by Dieudonné's theorem. ∎

As a corollary of the above theorem we have the corresponding topological result due to Künzi and Ferrario [6]:

COROLLARY 1. *The finest quasi–uniformity of each quasi–pseudometrizable topological space is bicomplete.*

A quasi–metric d on a set X is called bicomplete if d^* is a complete metric on X. The Sorgenfrey line, the Kofner plane and the Pixley–Roy space are relevant examples of nonmetrizable spaces which admit a compatible bicomplete quasi–metric.

Our next result, which extends Čech's theorem cited above, is contained in [8].

THEOREM 2. *A quasi–metrizable space* (X,P,Q) *has a compatible bicomplete quasi–metric if and only if* $(X,P\vee Q)$ *is a Čech complete space.*

Proof. We sketch the proof. If (X,P,Q) admits a compatible bicomplete quasi–metric d, then d^* is a complete metric on X compatible with $P\vee Q$.

Conversely, assume that d is a quasi–metric on X compatible with (P,Q) and let p be a complete metric on X compatible with $P\vee Q$. For each $n\in\mathbb{N}$, the set $G_n=\{(x,y):p(x,y)<2^{-n}\}$ is $(P\vee Q)\times(P\vee Q)$–open, so there is, for each $x\in X$, a sequence $<r_n(x)>$ of positive numbers such that $5r_{n+1}(x)<r_n(x)<2^{-n}$ and $U_n=\cup\{B_{d^*}(x,r_n(x))\times B_{d^*}(x,r_n(x)):x\in X\}\subseteq G_n$ for all $n\in\mathbb{N}$. Now put $V_n=\cup\{B_{d^{-1}}(x,r_n(x)/3)\times B_d(x,r_n(x)/3):x\in X\}$. Then $(V_n\cap V_n^{-1})\subseteq U_n$ and $V_{n+1}^3\subseteq V_n$ for all $n\in\mathbb{N}$. Hence, there is a quasi–pseudometric q on X such that $V_{n+1}\subseteq\{(x,y):q(x,y)<2^{-n}\}\subseteq V_n$ for all $n\in\mathbb{N}$. Finally, one can show that q is actually a bicomplete quasi–metric on X compatible with (P,Q). ∎

EXAMPLE. Let τ denote the usual topology on the set $\mathbb{Q}$ of rationals and let D be the discrete topology on $\mathbb{Q}$. It is well–known that $(\mathbb{Q},\tau)$ does not admit a compatible complete metric. However, Theorem 2 shows that $(\mathbb{Q},\tau,D)$ admits a compatible bicomplete quasi–metric because it is a quasi–metrizable space such that $(\mathbb{Q},D)$ is Čech complete.

The classical Niemytzky–Tychonoff theorem states that a metrizable space (X,T) is compact if and only if every metric on X compatible with T is complete. Quasi–metric extensions of this theorem may be found in [4],[7] and [10] for different notions of quasi–metric completeness. However, the cofinite topology on the set $\mathbb{N}$ of natural numbers provides an example of a compact quasi–metrizable topology which admits nonbicomplete compatible quasi–metrics. Our final result characterizes those quasi–metrizable spaces for which every compatible quasi–metric is bicomplete. The proof is given in [9].

THEOREM 3. *Let* (X,T) *be a quasi–metrizable space. Then, every quasi–metric on* X *compatible with* T *is bicomplete if and only if* X *is a finite set.*

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