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A Vectorial Expression for Liapounov's Central Limit Theorem

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In this paper we prove two Liapounov's central limit theorems for a sequence of independent p-dimensional random vectors, with mean and variance and covariance matrix Σ_n, in cases of both general and uniformly bounded sequence. Let $\{X_n\}$ be a sequence of p-dimensional random vectors, Varadarajan [2] proves that in order that the distributions of the X_n should be convergent in law to a limit, it is necessary and sufficient that the distribution of $l(X_n)$ should converge in law to some limit for every linear function l. In the next lemma we state a result about the limit law.

LEMMA 1. *Let $\{X_n\}$ be a sequence of p-dimensional random vectors. A necessary and sufficient condition for $X_n \xrightarrow{\mathcal{L}} X$ is that $c^T X_n \xrightarrow{\mathcal{L}} c^T X$ for each vector $c \in \mathbb{R}^p$.*

Proof. Let $\alpha_{X_n}(t)$ and $\alpha_X(t)$ be the characteristic functions of X_n and X. If $X_n \xrightarrow{\mathcal{L}} X$ then $\alpha_{X_n}(t)$ converges pointwise to $\alpha_X(t)$, so for each $c \in \mathbb{R}^p$ we have that the characteristic functions of $Y_n = c^T X_n$ and $Y = c^T X$ verify:

$$\alpha_{Y_n}(s) = \mathrm{E}[\exp(\mathrm{i} s Y_n)] = \mathrm{E}[\exp(\mathrm{i} s c^T X_n)] = \alpha_{X_n}(sc) \longrightarrow \alpha_X(sc) = \alpha_Y(s)$$

and then by the continuity theorem this implies that $Y_n = c^T X_n \xrightarrow{\mathcal{L}} Y = c^T X$. Conversely, if we assume that $c^T X_n \xrightarrow{\mathcal{L}} c^T X$ for each $c \in \mathbb{R}^p$, then taking $Y_n = t^T X_n$ and $Y = t^T X$, we have that:

$$\alpha_{X_n}(t) = \mathrm{E}[\exp(\mathrm{i} t^T X_n)] = \mathrm{E}[\exp(\mathrm{i} Y_n)] = \alpha_{Y_n}(1) \longrightarrow \alpha_Y(1) = \alpha_X(t)$$

hence $X_n \xrightarrow{\mathcal{L}} X$; this completes the proof. ∎

In the following theorem we assume that $\|\cdot\|$ is a norm such that $\|AB\| \leqslant \|A\| \, \|B\|$ for the product of matrices A and B.

LIAPOUNOV'S VECTORIAL THEOREM. *Let $\{X_n\}$ be a sequence of p-dimensional random vectors, with zero mean and variance and covariance matrix Σ_n, and let $\{a_n\}$ be a positive divergent sequence such that*

$$a_n^{-1}(\Sigma_1+\ldots+\Sigma_n) \longrightarrow \Sigma \quad \text{and} \quad a_n^{-1-\delta/2} \sum_{k=1}^{n} \mathrm{E}[\|X_k\|^{2+\delta}] \longrightarrow 0$$

for a positive δ, then:

$$\frac{X_1+\ldots+X_n}{\sqrt{a_n}} \xrightarrow{\mathcal{L}} \mathcal{N}_p(0,\Sigma).$$

Proof. Let Y_n be the random vector sequence:

$$Y_n = \frac{X_1+\ldots+X_n}{\sqrt{a_n}}$$

and let $c \in \mathbb{R}^p$ be any constant p-dimensional vector, then:

$$c^T Y_n = \frac{1}{\sqrt{a_n}} \sum_{k=1}^{n} U_k \tag{1}$$

where $U_k = c^T X_k$, $k=1,2,3,\ldots$ is a sequence of independent random variables, with zero mean, such that:

$$\begin{aligned} a_n^{-1-\delta/2} \sum_{k=1}^{n} \mathrm{E}[|U_k|^{2+\delta}] &\leq a_n^{-1-\delta/2} \sum_{k=1}^{n} \mathrm{E}[\|c\|^{2+\delta}\|X_k\|^{2+\delta}] = \\ &= \|c\|^{2+\delta} a_n^{-1-\delta/2} \sum_{k=1}^{n} \mathrm{E}[\|X_k\|^{2+\delta}] \longrightarrow 0 \end{aligned} \tag{2}$$

and whose variances satisfy:

$$s_n^2 = \sum_{k=1}^{n} \mathrm{Var}(U_k) = \sum_{k=1}^{n} c^T \Sigma_k c = a_n c^T \frac{\Sigma_1+\ldots+\Sigma_n}{a_n} c \tag{3}$$

$$\frac{s_n^2}{a_n} = c^T \frac{\Sigma_1+\ldots+\Sigma_n}{a_n} c \longrightarrow c^T \Sigma c. \tag{4}$$

If $c^T \Sigma c > 0$ then we have from (2) and (4) that:

$$\begin{aligned} s_n^{-2-\delta} \sum_{k=1}^{n} \mathrm{E}[|U_k|^{2+\delta}] &= \frac{a_n^{1+\delta/2}}{s_n^{2+\delta}} \cdot \frac{1}{a_n^{1+\delta/2}} \cdot \sum_{k=1}^{n} \mathrm{E}[|U_k|^{2+\delta}] \longrightarrow \\ &\longrightarrow (c^T \Sigma_k c)^{-1-\delta/2} \cdot 0 = 0 \end{aligned}$$

thus Liapounov's theorem (see pp. 275–277 of Loeve's book [1] for example) implies that:

$$\frac{U_1+\ldots+U_n}{s_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

and hence

$$c^T Y_n = \frac{U_1 + \ldots + U_n}{\sqrt{a_n}} = \frac{s_n}{\sqrt{a_n}} \cdot \frac{U_1 + \ldots + U_n}{s_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0, c^T \Sigma c). \tag{5}$$

On the other hand, if $c^T \Sigma c = 0$ then we have from (1) and (4) that:

$$\mathrm{E}[c^T Y_n] = 0 \quad \text{and} \quad \mathrm{Var}[c^T Y_n] = \frac{s_n^2}{a_n} \longrightarrow c^T \Sigma c = 0 \tag{6}$$

from which we obtain:

$$c^T Y_n \xrightarrow{\mathcal{P}} 0 \equiv \mathcal{N}(0, c^T \Sigma c). \tag{7}$$

Hence from (5) and (7) we have that:

$$c^T Y_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, c^T \Sigma c) \tag{8}$$

for each $c \in \mathbb{R}^p$. From lemma 1 and (8) we may obtain the conclusion of the theorem. ∎

LIAPOUNOV'S VECTORIAL THEOREM (bounded case). *Let $\{X_n\}$ be a sequence of independent and uniformly bounded p-dimensional random vectors, with zero mean and variance and covariance matrix Σ_n, and let $\{a_n\}$ be a positive divergent sequence such that $a_n^{-1}(\Sigma_1 + \ldots + \Sigma_n) \longrightarrow \Sigma$, then:*

$$\frac{X_1 + \ldots + X_n}{\sqrt{a_n}} \xrightarrow{\mathcal{L}} \mathcal{N}_p(0, \Sigma).$$

Proof. Let us denote by $\|\cdot\|$ the euclidean norm. Since $\|X_k\| \leqslant M < +\infty$, it follows that:

$$\begin{aligned} \mathrm{E}[\|X_k\|^{2+\delta}] &\leqslant M^\delta \mathrm{E}[\|X_k\|^2] = M^\delta \mathrm{E}[\mathrm{Tr}(X_k^T X_k)] = \\ &= M^\delta \mathrm{Tr}(\mathrm{E}[X_k X_k^T]) = M^\delta \mathrm{Tr}(\Sigma_k). \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} a_n^{-1-\delta/2} \textstyle\sum_{k=1}^n \mathrm{E}[\|X_k\|^{2+\delta}] &\leqslant M^\delta a_n^{-1-\delta/2} \textstyle\sum_{k=1}^n \mathrm{Tr}(\Sigma_k) = \\ = M^\delta a_n^{-\delta/2} \mathrm{Tr}[a_n^{-1}(\Sigma_1 + \ldots + \Sigma_n)] &\longrightarrow \frac{M^\delta}{\infty} \mathrm{Tr}(\Sigma) = 0, \end{aligned} \tag{10}$$

and Liapounov's vectorial theorem applies. This concludes the proof. ∎

REFERENCES

1. LOÈVE, M., "Probability Theory", 3rd ed., Van Nostrand Reinhold Co., New York, 1963.

2. VARADARAJAN, V.S., A useful convergence theorem, *Sankhyā* **20** (1958), 221–222.