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Extraction of Subsequences in Banach Spaces

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THE PROPERTIES

Surely, one of the earliest *extraction-of-subsequences property* is the Riesz characterization of finite-dimensional Banach spaces:

(F) *Each bounded sequence admits a convergent subsequence.*

In the search of an infinite-dimensional analogue of (F), the following weaker form of (F) appeared:

(W) *Each bounded sequence admits a weakly convergent subsequence.*

The Banach-Alaglou and Eberlein-Smullian theorems assert that property (W) is equivalent to the reflexivity of the space. Rosenthal's ℓ_1-theorem [38] is a *tour de force* based on this idea: if "weakly convergent" is replaced by "weakly Cauchy" one obtains an equivalent condition for *"X does not contain a copy of ℓ_1"*. We shall not enter into this line of results. Recently, Rosenthal [39] obtained an *extraction-of-subsequences* property equivalent to the property of *"not containing a copy of c_0"*:

(Nc_0) *Each weak-Cauchy sequence admits a basic subsequence satisfying*

$$\sup_{n\in\mathbb{N}} \|\textstyle\sum_{j=1}^{n} c_j x_j\| < +\infty \quad \Rightarrow \quad \sum c_j \ \textit{converges}.$$

The property of not having quotients isomorphic to c_0 has been characterized by González and Onieva [21]:

(NQc_0) *Each weak* convergent sequence in X^* admits a weak Cauchy subsequence.*

There remain unknown characterizations of the containment of copies of other ℓ_p.

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Going back to our mainstream, an application of the Hahn–Banach theorem shows that if (x_n) is a weakly convergent sequence in a Banach space X, having the point x as its weak limit, then there exists a sequence (σ_n) formed by convex combinations of the x_n's, which is norm convergent to x. In fact, the sequence (x_n) is weakly null if and only if for each $\epsilon > 0$ there exists a finite convex combination $\Sigma_i \tau_i x_i$ such that $\|\Sigma_i \pm\tau_i x_i\| < \epsilon$ for every choice of signs. An old question going back to Banach and Saks [3] is whether it is possible that that convex combination can be chosen to be the arithmethic means; i.e., $\sigma_n = (x_1 + \ldots + x_n)/n$.

DEFINITION 1. A sequence (x_n) in a Banach space X is said to be a *Banach–Saks* sequence if it has norm convergent arithmetic means; i.e., if the sequence $(x_1 + \ldots + x_n)/n$ converges. A Banach space X is said to have the *Banach–Saks property* if

(BS) *Each bounded sequence admits a Banach–Saks subsequence.*

Examples of Banach spaces having and having not this property shall be shown later. Here we want to introduce several stronger forms of reflexivity and of the Banach-Saks property. To this end, we define two kinds of sequences in a Banach space:

DEFINITION 2. ([25]) A sequence (x_n) in a Banach space X is said to be p-Banach-Saks, $1 < p < +\infty$, if, for some constant λ_p and all $N \in \mathbb{N}$, $\|\Sigma_{k=1}^N x_k\| \leqslant \lambda_p \cdot N^{1/p}$. We shall denote by $\lambda_p(\{x_n\})$ the infimum of those constants λ_p satisfying the above inequality. The sequence (x_n) is said to be p-Banach-Saks convergent to x if the sequence $(x_n - x)$ is p-Banach-Saks.

Notice that a p-Banach-Saks sequence can contain subsequences having constant λ arbitrarily large, and even unbounded subsequences: the sequence (x_k) of real numbers taking the values $x_k = n$, for $k = 2^n$, and $x_k = 0$ otherwise is a simple example. We shall call *hereditarily p-Banach-Saks* to a sequence (x_n) such that, for some constant $\lambda > 0$, $\|\Sigma_{k=1}^N x_{n_k}\| \leqslant \lambda \cdot N^{1/p}$ for every choice of integers $n_1 < \ldots < n_N$. Hereditarily p-Banach-Saks sequences were introduced by Pelczynski [36] under the name of $\tau_{1/p}$-convergent sequences.

DEFINITION 3. A sequence (x_n) in a Banach space X is said to be *weakly-p-summable*, $1 \leqslant p < +\infty$, if $(x^*(x_n))_n \in \ell_p$ for each $x^* \in X^*$. Equivalently, if a constant $w_p > 0$ exists such that, for all $N \in \mathbb{N}$, $\|\Sigma_{k=1}^N \alpha_n x_n\| \leqslant w_p \cdot \|(\alpha_n)\|_{p^*}$. We shall denote by $w_p(\{x_n\})$ the infimum of those constants w_p satisfying the above inequality. The sequence (x_n) is said to be weakly-p-convergent to x if the sequence $(x_n - x)$ is weakly-p-summable.

Weakly-p-summable and p^*-Banach-Saks sequences have some similitude: a sequence (x_n) is weakly-p-summable when a bound of the form $\|\Sigma_{k=1}^N \theta_n x_n\| \leqslant K$ exists for all sequences (θ_n) belonging to the unit ball of ℓ_{p^*}; the sequence (x_n) is p^*-Banach-Saks when a bound of the form $\|\Sigma_{k=1}^N \theta_n x_n\| \leqslant$ $\leqslant K$ exists when (θ_n) is any of the following sequences in the unit ball of ℓ_{p^*}:

$$(1, 0, 0, \ldots)$$
$$(\frac{1}{\sqrt[p^*]{2}}, \frac{1}{\sqrt[p^*]{2}}, 0, 0, \ldots)$$
$$(\frac{1}{\sqrt[p^*]{3}}, \frac{1}{\sqrt[p^*]{3}}, \frac{1}{\sqrt[p^*]{3}}, 0, 0, \ldots)$$
$$\ldots$$

It is, therefore, obvious that weakly-p-summable sequences are p^*-Banach-Saks. In fact, weakly-p-summable sequences are even hereditarily-p^*-Banach-Saks. The converse is false in a strong sense. In $\mathbb{R}$, the sequence $(n^{-1/2})$ is hereditarily-2-Banach-Saks but not weakly-2-summable. Nevertheless, it contains, naturally, weakly-2-summable subsequences. We proceed to show an hereditarily-2-Banach-Saks sequence containing no weakly-2-summable subsequences:

EXAMPLE 4. Let Δ_n denote the subset of $[0,1]$ formed by all real numbers such that their n^{th} cipher, when represented in base 3, is 0 or 2. Now consider the following sequence of functions inductively defined:

$$f_1 = \chi_{\Delta_1} \qquad \text{and} \qquad f_{n+1} = (1 + \max\{f_1, \ldots, f_n\}) \cdot \chi_{\Delta_{n+1}} .$$

The sequence $g_n(t) = f_n^{-1/2}(t)$ when $f_n(t) \neq 0$, and $g_n(t) = 0$ otherwise, is normalized and hereditarily 2-Banach-Saks in $C[0,1]^{**}$, but no subsequence of it is weakly-2-summable.

Another aspect of the differences between both classes of sequences is the following: any weakly-2-summable, as the continuous image of the canonical basis of ℓ_2, lies inside the range of a vector measure (see [1]). There exist, however, 2-Banach-Saks sequences which do not lie inside the range of a vector measure, e.g.: the unit vector basis of the Lorentz sequence space constructed with norming sequence (c_n) defined by the equality $\Sigma_{n \leqslant N} c_n = n^{1/p}$ (see [37], [12]). The sequence of Example 4 is another example.

With these definitions we are ready to strenghen the Banach-Saks property:

DEFINITION 5. A Banach space X is said to have the p-Banach-Saks property $(1 < p < +\infty)$ if

(BS_p) *Each bounded sequence admits a p-Banach-Saks convergent subsequence.*

And now the reflexivity:

DEFINITION 6. We shall say that a Banach space $X \in W_p$, $1 \leqslant p < +\infty$, if

(W_p) *Each bounded sequence admits a weakly-p-convergent subsequence.*

To be able to work also with non-reflexive spaces, we shall speak about those properties in the *weak* sense when *weakly null* sequences admit a subsequence of the type indicated. Thus, a Banach space has the *weak-Banach-Saks* property if weakly null sequences admit subsequences having norm convergent arithmetic means. It has the *weak-*W_p property if weakly null sequences admit weakly-p-summable subsequences, and it has the *weak-p-Banach-Saks* property if weakly null sequences admit p-Banach-Saks subsequences.

It is clear that property (weak) W_p implies the (weak) p^*-Banach-Saks property. Surprisingly, the converse is almost true:

THEOREM 7. *Let X be a Banach space with the (weak) p-Banach-Saks property. Then, for all* $r > p^*$, *X has the (weak)* W_r *property.*

A direct proof of this result can be found in [9]. A proof of the implication weak-p-Banach-Saks $\Longrightarrow$ weak W_r, for all $r > p^*$ was provided by Rakov [36]. Here we show a third different method:

Proof. If (x_n) is a weakly-null sequence in a Banach space X, and $p > 1$ is given then, for every choice of positive integers i, j the set

$$F_{i,j} = \left\{ (n_k) \in P_\infty(\mathbb{N}) \,:\, \frac{\| \Sigma_{k=1}^N x_{n_k} \|}{N^{1/p}} \leqslant \frac{1}{i} \;\; \forall\, N > j \right\}$$

is closed in $P_\infty(\mathbb{N})$. Hence the set

$$B_p = \left\{ (n_k) \in P_\infty(\mathbb{N}) \,:\, \lim_{N\to\infty} \frac{\| \Sigma_{k=1}^N x_{n_k} \|}{N^{1/p}} \leqslant 0 \right\} = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} F_{i,j}$$

is a Borel subset of $P_\infty(\mathbb{N})$ and so completely Ramsey. Thus, if X has the p-Banach-Saks property then for each $q<p$, every bounded sequence in X has a subsequence which is hereditarily q-Banach-Saks convergent.

Now, it is not difficult to see that if (x_n) is a normalized sequence in a Banach space satisfying

$$\|\Sigma_{i=1}^N \lambda_i x_{k_i}\| \leqslant C \cdot N^{1/p} \cdot \max\{|\lambda_i| : 1 \leqslant i \leqslant N\}$$

for all sequences (λ_i) of scalars and (k_i) of integers, and for all integers N, then for any $1<r<p$ there exists a constant D such that

$$\|\Sigma_{i=1}^N \lambda_i x_{k_i}\| \leqslant D \cdot \|(\lambda_i)\|_r$$

(the first part of this proof was provided by the referee of [9]; the second part is the correct statement of Lemma 3.4 of [19]). ∎

Rakov [37], gave an example showing that weak-p-Banach-Saks does not imply weak W_{p^*}. We conjecture that the same is true in the general case:

OPEN PROBLEM. *Are W_p and p^*-Banach-Saks equivalent?*

Maybe it is interesting here to remark that Knaust [30] has proved the implication p-Banach-Saks $\Longrightarrow W_{p^*}$ for Orlicz sequence spaces.

Another surprising fact concerning subsequences was established by Knaust and Odell in [27] and [28]:

THEOREM 8. (Knaust and Odell) *In a Banach space X having the weak-W_p property, $1 \leqslant p < +\infty$, there exists a uniform constant $c_p(X)$ such that every weakly null sequence admits a weakly-p-summable subsequence having constant not greater than $c_p(X)$.*

The behaviour of the uniform constant c_p has some interest. In [10], it was shown that if (X_n) is a sequence of Banach spaces and λ is a Banach sequence space having a monotone basis then $c_p(\lambda(X_1, X_2, \dots)) \leqslant c_p(\lambda) + \sup\{c_p(X_n)\}$.

One of the most striking results about extraction of subsequences is the following one of Brunel and Sucheston [7].

THEOREM 9. *Each bounded sequence (x_n) in a Banach space X contains a subsequence (x_m) with the following property: for each finite set $I \subset \mathbb{N}$ and each finite sequence $a = (a_i)_{i \in I}$ there exists a number $L(a)$ such that*

$$L(a) = \lim \|\Sigma_{i=1}^{N} a_i\, x_{m(i)+f(i)}\| \qquad \text{as} \quad f(i) \to \infty .$$

Proof. Let $a = (a_i)_{i \in I}$ be a fixed sequence of rational numbers. Define a function $\psi : \mathbb{N}^{\operatorname{card} I} \longrightarrow \mathbb{R}^+$ by $\psi(\overline{n}) = \|\Sigma_{i=1}^{N} a_i\, x_{n_i}\|$. The image of ψ lies in some interval, say $[0, \alpha]$. Consider the subsets $A_1 = \{\overline{n} : \psi(\overline{n}) \in [0, \alpha/2]\}$ and $B_1 = \{\overline{n} : \psi(\overline{n}) \in [\alpha/2, \alpha]\}$. Obviously, A_1 or B_1 is infinite. Given any sequence (i_n) of integers, Ramsey's theorem allows one to conclude that either A_1 or B_1 contains all $\overline{n}$ formed from terms of a certain subsequence of (i_n). Iterating the argument and diagonalizing the subsequences obtained, one arrives to a point $L(a)$, intersection of a nested sequence of closed intervals, such that $|\psi(\overline{n}) - L(a)| < \epsilon$ for any choice $n_1 < \ldots < n_N$ sufficiently far in the final subsequence.

A new diagonalization passing through an enumeration of all finite sequences of rationals leads to a new subsequence with the desired property.

The case of arbitrary finite sequences, not necessarily rationals, follows by density. ∎

This result of Brunel and Sucheston is the key behind Rakov's approach to the p-Banach-Saks property [37]:

THEOREM 10. *Let X be a Banach space with the Banach-Saks property. Assume that each weakly null sequence admits, for some $p > 1$, a p-Banach-Saks subsequence. Then there exists some number $q > 1$ such that each weakly null sequence admits a q-Banach-Saks subsequence.*

THE EXAMPLES

Clearly, ℓ_p spaces, $1 < p < +\infty$, have the W_{p^*} property, while ℓ_1 and c_0 have the weak-W_1 property. L_p spaces have property $W_{\max\{2,p^*\}}$. Szlenk [43], proved that $L_1[0,1]$ has the weak-Banach-Saks property. Since $L_1[0,1]$ contains isomorphic copies of ℓ_p, $1 \leqslant p \leqslant 2$, it is clear that it does not have the p-Banach-Saks property for no $p > 1$. Rakov's result then implies that Szlenk's result is optimal.

The first example of a Banach space not having the weak-Banach-Saks property was constructed by Schreier [42]. This is the space obtained by completion of the space of finite sequences with respect to the following norm:

$$\|x\|_S = \sup_{[A \text{ admissible}]} \Sigma_{j \in A} |x_j| ,$$

where a finite sub-set of natural numbers $A = \{n_1 < \ldots < n_k\}$ is said to be *admissible* if $k \leqslant n_1$.

Kakutani [26], proved that a uniformly convex Banach space has the Banach–Saks property. Later on, Nishiura and Waterman [34] proved that a Banach space with the Banach–Saks property must be reflexive, a fact which led Sakai [40] to ask whether a reflexive Banach space must also have the Banach–Saks property. Baerstein [2] showed a method to obtain, for each $1 < p < +\infty$, a reflexive Banach space B_p which does not have the Banach–Saks property. These are the spaces obtained by completion of the space of finite sequences with respect to the following norm:

$$\|x\|_{B_p} = \sup\left\{\left[\Sigma_k \|E_k\|_1^p\right]^{1/p} : E_1 < E_2 < \ldots < E_n\,,\ n \in \mathbb{N}\right\},$$

where each E_n is admissible.

Beauzamy and Lapresté [4], remarked that since the canonical inclusion $\ell_1 \hookrightarrow S$ is weakly compact but not a Banach–Saks operator, real interpolation between ℓ_1 and S would provide other examples of reflexive Banach spaces without the Banach–Saks property.

Recovering Kakutani's ideas, it is possible to prove [9]:

THEOREM 11. *Let X be an infinite dimensional super–reflexive Banach space. Then there are numbers $1 < q < p$ such that $X \in W_p$ and $X \notin W_q$.*

Since any ℓ_p-sum of finite dimensional spaces is in W_{p^*} but not in W_r, for all $r < p^*$, the space $\Sigma_p\, \ell_1^k$ shows that Theorem 11 is not an equivalence. Tsirelson's space is an example of a non super–reflexive Banach space having all W_p properties $(1 < p)$. See [11] for details. A similar result to this seems to have been obtained by Knaust [31] and Kotlyar [33].

The calculus of p could be: let δ be such that any two points in the unit sphere such that $\|x - y\| \geqslant \epsilon$ satisfy $\|x + y\| \leqslant 2(1 - \delta)$. Take p such that $(2(1 - \delta))^p < 2$. The space is W_{p^*}. Kotlyar calculates p as $p = \log_{2/\theta}(1/\theta)$ where $\theta = \min_{0<\epsilon<1} \max\{(\epsilon + 1)/2\,,\ 1 - \delta(\epsilon)\}$ and $\delta(\epsilon)$ is the modulus of convexity.

Given a property P of Banach spaces, a second set of problems which naturally arises is whether "X has property P" implies that $L_p(X)$ or $\ell_p(X)$ have property P. In [35], Partington proved that the Banach–Saks property passes from X to $\ell_p(X)$; in fact, he proved:

THEOREM 12. *Let λ be a Banach sequence space having a monotone basis, and let (X_n) be a sequence of Banach spaces. If each X_n and λ have the Banach-Saks property, then the same is true for $\lambda(X_1, X_2, \ldots)$.*

The result is false if "Banach-Saks" is replaced by "weak-Banach-Saks": a careful choice $1 < p_n < \log n / (\log n - \log c)$ implies that the sequence $x_n = (e_n, e_n, e_n, \ldots)$ (n times) does not contain Banach-Saks subsequences in the space $c_0(\ell_{p_1}, \ell_{p_2}, \ldots, \ell_{p_n}, \ldots)$.

It can be proved that Partington's result is "almost" true for the p-Banach-Saks properties. Firstly, one verifies that W_p properties pass to vector sequence spaces [10]:

THEOREM 13. *Let λ be a Banach sequence space having a monotone basis, and let X be a Banach space. If X and λ have the property (weak) W_p, then the same is true for $\lambda(X)$.*

and then, using the implication p-Banach-Saks $\Longrightarrow W_r$ for all $r > p^*$, one obtains:

If λ and X have the (weak) p-Banach-Saks property, then $\lambda(X)$ has the (weak) r-Banach-Saks property for all $r < p$.

The passage to $L_p(X)$ seems to be more difficult. Schachermayer [41] and Bourgain [22] showed that the Banach-Saks property is not L_2-hereditary. Schachermayer produced a tree-like version of Scherier space, which we shall denote E. Since the canonical inclusion $\ell_1 \hookrightarrow E$ is a Banach-Saks operator, the real interpolation spaces $E_{\theta,q} = (\ell_1, E)_{\theta,q}$ have the Banach-Saks property. Schachermayer proved that $L_2([0,1], E_{\theta,q})$ has not the Banach-Saks property. Beauzamy and Lapresté [4], used a tree-like version of Baernstein spaces, E_p, to show that the Banach-Saks property is not L_2-hereditary. It can also be seen that E_p has property W_{p^*}, from which it follows that properties W_p are not L_p-hereditary.

Despite these examples of Schachermayer and Bourgain, Cembranos has obtained the following remarkable result [15].

THEOREM 14. *For a Banach space X the following conditions are equivalent:*

(1) *$L_1(X)$ has the weak-Banach-Saks property.*

(2) *$L_p(X)$ has the weak-Banach-Saks property $(1 \leqslant p < +\infty)$.*

(3) $L_1(X)$ *is weak Komlós.*

From which it follows that the following conditions are also equivalent:

(i) $L_1(X)$ *has the weak-Banach-Saks property and* X *is reflexive.*

(ii) $L_p(X)$ *has the Banach-Saks property* $(1 < p < +\infty)$.

(iii) $L_1(X)$ *is Komlós.*

Recall here Komlós' theorem:

THEOREM 15. *For every bounded sequence* (f_n) *in* $L_1(\mu)$ *there exist a subsequence* (f_m) *and* f *in* $L_1(\mu)$ *such that*

$$\frac{1}{N}\Sigma_{i=1}^{N} f_{m_i} \longrightarrow f \quad \textit{almost everywhere}$$

for each subsequence (f_{m_i}) *of* (f_m).

When Komlós theorem holds in the space $L_1(X)$ it is said that $L_1(X)$ is a Komlós space. When Komlós theorem holds replacing "bounded sequence" by "weakly null sequence" it will be said that $L_1(X)$ is a weak Komlós space.

THE APPLICATIONS

Hereditary Dunford-Pettis property. A Banach space X is said to have the Dunford-Pettis property if weakly compact operators defined on X are compact or, equivalently, if for arbitrary weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, $\lim <x_n^*, x_n> = 0$. A Banach space X is said to have the hereditary Dunford-Pettis property if each closed subspace of X has the Dunford-Pettis property.

Using a result of Elton [16], it can be seen that property weak W_1 is equivalent to the hereditary Dunford-Pettis property and, therefore, to property S of Knaust and Odell [27]. One has:

THEOREM 16. *If* X *and* λ *have the hereditary Dunford-Pettis property, then* $\lambda(X)$ *has the hereditary Dunford-Pettis property.*

This includes the cases $\lambda = c_0, \ell_1$ of Cembranos [13], [14] and $\lambda = \ell_1$ of Knaust [29].

Spaces of Polynomials. Let X be a (real or complex) Banach space. For $m = 1, 2, \dots$, a map P is said to be a continuous m-homogeneous polynomial if it has the form $P(x) = A(x, x, \dots, x)$, where A is a continuous m-linear scalar valued form on X^m. A continuous polynomial is a finite sum

$P = P_0 + P_1 + \ldots + P_n$, where P_0 is a constant and each P_m is a continuous m-homogeneous polynomial. A sequence $(x_n) \subset X$ is said to be weak-polynomial convergent to x if $P(x_n) \to P(x)$ for all continuous polynomials on X. In [8], Carne, Cole and Gamelin defined a Λ-space as a Banach space such that every sequence weak-polynomial convergent to zero is norm null, and conjectured that, for $1 < p < +\infty$, L_p-spaces are Λ-spaces, proving themselves the conjecture for $p \geqslant 2$. In [23], Jaramillo and Prieto proved that super-reflexive spaces are Λ-spaces. In fact, what they proved is:

THEOREM 17. *If X^* has, for some $p < +\infty$, property W_p, then X is a Λ-space.*

which is a combination of Theorem 11 and the observation that a weakly-p-summable sequence (x_n^*)in X^* defines a continuous N-homogeneous polynomial $P(x) = \Sigma_{k=1}^{\infty} \left(x_k^*(x)\right)^N$ in X.

This result allows one to prove that several interesting Banach spaces are Λ-spaces: Tsirelson space T (the dual of Tsirelson original space, since $T^* \in W_p$, for all $p > 1$; see [11]), the duals of Schachermayer spaces $(E_q)^*$ (since $E_q \in W_{q^*}$; see [10] and [4]) and Baernstein B_q spaces:

THEOREM 18. *Let $1 < p < +\infty$. Baernstein's spaces B_p are Λ-spaces.*

Proof. Following Theorem 12, it is only needed to verify that $(B_p)^*$ has property W_p.

Let (u_k) be a bounded sequence of $(B_p)^*$. By the reflexivity of $(B_p)^*$ and a standard application of the Besaga-Pelczynski selection principle, it can be assumed that (u_k) is a weakly null sequence of $(B_p)^*$ formed by normalized blocks of the canonical basis (e_n^*), i.e., $u_k = \Sigma_{n \in I_k} \lambda_n x_n$, for some sequence $I_1 < I_2 < \ldots$ of finite sets. This sequence of blocks is weakly-p-summable:

$$\|\Sigma_{j=1}^{N} \theta_j u_j\|_{(B_p)^*} = \sup\{|\Sigma_{j=1}^{N} \theta_j u_j(x)| : \|x\|_{B_p} \leqslant 1\} \leqslant$$

$$\sup\{\Sigma_{j=1}^{N} |\theta_j u_j(x)| : \|x\|_{B_p} \leqslant 1\} \leqslant \sup\{\Sigma_{j=1}^{N} |\theta_j| \, \|I_j x\|_{B_p} : \|x\|_{B_p} \leqslant 1\} \leqslant$$

$$\sup\left\{ \left[\Sigma_{j=1}^{N} |\theta_j|^{p^*}\right]^{1/p^*} \left[\Sigma_{j=1}^{N} \{\|I_j x\|_{B_p}\}^p\right]^{1/p} : \|x\|_{B_p} \leqslant 1 \right\}.$$

Given $\epsilon > 0$, it is clear that there is a chain of admissible sets $\{E_j\}$ such that $\{\|I_j x\|_{B_p}\}^p \leqslant \{\|E_j x\|_1\}^p + \epsilon/2^j$. This finishes the proof. ∎

Comment. Seifert proved that $(B_p)^*$ has the Banach-Saks property.

For every $1 < p < +\infty$, $\ell_p(T)$, $\ell_p(B_q)$ or $\ell_p((E_q)^*)$ are also Λ-spaces.

Other results where a suitable extraction of subsequences pays off can be found in [6] and [20].

THEOREM 19. ([20]) *If X^* has, for some $1 < p < +\infty$, the property W_p, then the space of all weakly continuous m-homogeneous polynomials $P_w(^m X)$, $m \geqslant p$, contains an isomorphic copy of c_0.*

and

THEOREM 20. ([6]) *If X^* has, for some $1 < p < +\infty$, the property W_p, then every R-bounding set (i.e., every set where all rational functions are bounded) is relatively compact in X.*

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