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Characterization of Linear Rational Preference Structures

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1. INTRODUCTION

We consider the multiobjective decision making problem. The decision maker's (DM) impossibility to take consciously a preference or indifference attitude with regard to a pairs of alternatives leads us to what we have called doubt attitude. So, the doubt may be revealed in a conscient way by the DM. However, it may appear in an inconscient way, revealing judgements about her/his attitudes which do not follow a certain "logical reasoning".

In this paper, doubt will be considered as a part of the information revealed by the DM. In the next paragraphs we will see what we mean with the term "logical reasoning" introducing the rational preference structure concept.

2. THE RATIONAL PREFERENCE STRUCTURE

From now, we will denote by $Y \subset \mathbb{R}^n$ the objective or consequence space in a multiobjective decision making problem.

DEFINITION 2.1. A rational preference structure on Y is a pair of binary relations on such set, denoted by (R_1,R_2), that fulfills the following axioms

E1: R_1 is asymmetric and transitive. R_1 is called rational preference on Y.

E2: R_2 is an equivalence relation. R_2 is called rational indifference on Y.

E3: R_1 and R_2 are disjoints.

E4: If $(y^1,y^2) \in R_1$ and $(y^2,y^3) \in R_2 \Rightarrow (y^1,y^3) \in R_1$ for every y^1, y^2, $y^3 \in Y$.

E5: If $(y^1,y^2) \in R_2$ and $(y^2,y^3) \in R_1 \Rightarrow (y^1,y^3) \in R_1$ for every y^1, y^2, $y^3 \in Y$.

The rational preference structure is not a tool to describe how a DM takes a decision, but how it should be taken by a rational individual. This definition was introduced in [3] and follows the approach considered in [2], and more recently

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in [1].

If we wish to work with a concept in which it appears the four attitudes revealed by a DM in her/his judgements about paired comparisons, we would have to consider next definition.

DEFINITION 2.2. Let (R_1,R_2) be a rational preference structure on Y. A quatern of binary relations associated to (R_1,R_2), is the set of binary relations (R_1,R_1^s,R_2,R_{12}^c), where R_1^s is the symmetrical of R_1 on $Y\times Y$ with regard to the diagonal Δ, and R_{12}^c is the complement of $R_1 \cup R_1^s \cup R_2$ on $Y\times Y$. The latter is called doubt.

There is a difference between the attitudes which reveals the DM to the analyst and the binary relations which model these relations by means of the concept of quatern associated to a rational preference structure. Such difference appears in R_{12}^c because contains, besides the doubt revealed by the DM, those pairs where axioms E4 – E5 are not fulfilled (conscient doubt). The way in which "the doubt is revealed by the DM" to R_{12}^c is given in the following general result.

THEOREM 2.1. *Given two binary relations R and S on Y such that the diagonal Δ on $Y\times Y$ is contained in S, and let C_i be the maximal subset of $R\cup S \subset Y\times Y$ which fulfills the following properties:*

1) $R_{1i}^* = R\cap C_i$ *is asymmetric and transitive.*
2) $R_{2i}^* = S\cap C_i$ *is an equivalence relation.*
3) *On C_i the relations R_{1i}^* and R_{2i}^* fulfill axioms* E4 *and* E5.

Then $(R_{1i}^* \setminus R_{2i}^*, R_{2i}^*)$ is a rational preference structure.

This rational preference structure will be called rational preference structure associated to R,S and C_i, and will be denoted by $(R_1,R_2)_{R,S,C_i}$. We define a rational preference structure associated to R and S, as $(R_1,R_2)_{R,S,C}$, where $C = \cap C_i$ (see [4]).

The rational preference structure concept though rigorously models the different attitudes, it does not constitute by its own a good tool to represent such attitudes and so, to look for the efficient solutions, which is an essential purpose in multiobjective problems. Then, it seems reasonable to restrict to those preference structures which can be analytically represented by means of a family of functions.

DEFINITION 2.3. Let (R_1,R_2) be a preference structure on $Y \subset \mathbb{R}^n$ and V a family of functions of class $C^k(\mathbb{R}^n)$. We say that (R_1,R_2) is a V-rational preference of class k, if $(R_1,R_2) = (R_1,R_2)_{R,S}$, where

$$(y^1,y^2) \in R \iff \nu(y^1) \geqslant \nu(y^2) \quad \forall \nu \in V$$

$$(y^1,y^2) \in S \iff \nu(y^1) = \nu(y^2) \quad \forall \nu \in V$$

Furthemore, if V is a family of linear functions, we say that (R_1,R_2) is a linear rational preference structure.

3. LINEAR APPROXIMATION STRUCTURE

Based on the general concepts of global preferred, dominated, indifferent and doubt direction cones, which can be found in [1], we shall propose the next definition as a translation of such concepts to the rational preference structure frame.

DEFINITION 3.1. Let (R_1,R_2) be a rational preference structure on $Y \subset \mathbb{R}^n$.

1. The vector $d \in \mathbb{R}^n$ is a global preferred, dominated, indifferent or doubt direction for $y^0 \in Y$, if for every $\alpha > 0$, $(y^0 + \alpha d, y^0)$ belongs to R_1, R_1^s, R_2 or R_{12}^c, respectively.

The collection of all global preferred, dominated, indifferent and doubt directions for y^0, are respectively called the global preferred, dominated, indifferent and doubt cone for y^0, and will be denoted by $P(y^0)$, $D(y^0)$, $I(y^0)$ and $DD(y^0)$, according to the case.

2. The vector $d \in \mathbb{R}^n$ is a local preferred, dominated, indifferent or doubt direction for y^0, if there is $\alpha_0 > 0$ ($\alpha_0 \in \mathbb{R}$ and fixed) such that, whenever $0 < \alpha < \alpha_0$, $(y^0 + \alpha d, y^0)$ belongs to R_1, R_1^s, R_2 or R_{12}^c, respectively.

Analogously we will have the local preferred, dominated, indifferent and doubt cone for y^0, and will be denoted by $LP(y^0)$, $LD(y^0)$, $LI(y^0)$ and $LDD(y^0)$, respectively.

Let us now see what we understand by linear approximation preference structure.

DEFINITION 3.2. Let (R_1,R_2) be a rational preference structure on $Y \subset \mathbb{R}^n$. The lower linear approximation structure is the quatern of binary relations (L_1, L_2, L_3, L_4) on $\mathbb{R}^n$ defined by

$$L_i = \cup\{(y^0+\alpha d, y^0) \ : \ y^0 \in Y, \ d \in K_i(y^0), \ \alpha > 0\} \quad i = 1,2,3,4,$$

where $K_i(y^0)$, $i = 1,2,3,4$, is equal to $P(y^0)$, $D(y^0)$, $I(y^0)$ and $DD(y^0)$, respectively.

If the cones $K_i(y^0)$, $i = 1,2,3,4$, were defined by those directions $d \in \mathbb{R}^n$ such that exists $\alpha > 0$ satisfying that $(y^0+\alpha d, y^0)$ belongs to R_1, R_1^s, R_2 or R_{12}^c, respectively, we would obtain the upper linear approximation structure, denoted by (L_1', L_2', L_3', L_4').

In these definitions, we have used the "linear" term. This is due to the fact that satisfies the following property

$$(y^1,y^2) \in R \implies (y^2+\alpha(y^1-y^2),y^2) \in R \ \ \forall \alpha > 0 \qquad \text{(P)}$$

Let us now consider a result, whose proof is in [5], which characterizes the lower and upper approximation preference structures.

THEOREM 3.1. *Let (R_1,R_2) be a rational preference structure on $\mathbb{R}^n$ and $(S,\ll)$ the ordered set of associated binary quaterns on $\mathbb{R}^n$, whose order is defined by*

$$(A,B,C,D) \ll (A',B',C',D') \iff A \subset A', \, B \subset B', \, C \subset C', \, D \subset D'.$$

Let us consider the sets

$$C = \{(S_1,S_2,S_3,S_4) \in S : S_i \text{ satisfies (P)}, \ i = 1,2,3,4, \text{ and } (S_1,S_2,S_3,S_4) \ll (R_1, R_1^s, R_2, R_{12}^c)\}$$

$$C' = \{(S_1,S_2,S_3,S_4) \in S : S_i \text{ satisfies (P)}, \ i = 1,2,3,4, \text{ and } (R_1, R_1^s, R_2, R_{12}^c) \ll (S_1,S_2,S_3,S_4)\}$$

then

$$(L_1,L_2,L_3,L_4) = \max C \quad \text{and} \quad (L_1', L_2', L_3', L_4') = \min C'.$$

4. CHARACTERIZATION OF LINEAR RATIONAL REFERENCE STRUCTURES FROM THE LINEAR APPROXIMATION STRUCTURES

Our purpose is, given the linear approximations to V–rational preference, deduce from them if we have or not a linear V–rational preference.

LEMMA 4.1. *The quatern of binary relations associated to a linear rational preference structure (R_1,R_2) fulfills the linearity property (P).*

The next theorem provides a characterization of the linear rational preference structures from their linear approximation structures.

THEOREM 4.2. *Let* (R_1,R_2) *be a rational preference structure on* $\mathbb{R}^n$. (R_1,R_2) *is a linear rational preference structure if and only if*

1) $L_1 = L_1'$ *and* $L_3 = L_3'$.

2) L_i, $i = 1,3$ *is compatible with addition on* $\mathbb{R}^n$, *that is*

$$(y^1,y^2) \in L_i \implies (y^1+y,y^2+y) \in L_i, \ \forall y \in \mathbb{R}^n, \ i = 1,3.$$

5. CONCLUSIONS

The linear approximation structures to a rational preference are easy concepts to obtain in practice, from the interaction between the analyst and the DM. This ease is complemented by the usefulness of this tool to describe preference structures represented by family of functions. In this paper we have only considered the representation by means of families of linear functions. However, an open problem in this context, would be to describe, from these approximations, families of functions under more complex analytical conditions.

REFERENCES

1. CHIEN, I.S., YU, P.L., ZHANG, D., Indefinite preference structures and decision analysis, *J. Optim. Theory Appl.* **64** (1990), 71–86.
2. FRENCH, S., "Decision Theory: An Introduction to the Mathematics of Rationality", Horwood, New York, 1986.
3. PACHÓN, J.G., Aproximaciones de estructuras de preferencia en problemas de decisión multiobjetivo, Ph. D. Thesis, Univ. Politec. Madrid, 1990.
4. PACHÓN, J.G., Un modelo normativo para las preferencias en problemas de decision multiobjetivo, submitted to *Rev. R. Acad. Ciencias.*
5. PACHÓN, J.G., RÍOS-INSUA, S., Linear approximation structures to preference structures, working paper 1/1991, Dpt. A.I., Politechnical University of Madrid.

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