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Partially Flat and Projective Modules[1]

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Assume that A and B are rings with identity and that M and N are (B,A) and (A,B) bimodules, respectively. We say that ${}_AN$ (respt. M_A) is *partially flat* (respt. *projective*) *with respect to a subcategory* $\mathscr{C}(A)$ *of* **Mod**$-A$ (the category of all unital right A-modules), if the tensor functor, $-\otimes_A N$ (respt. the hom functor $\mathrm{Hom}_A(M,-)$) is exact on $\mathscr{C}(A)$. For example, a flat or a projective module is partially flat or projective with respect to **Mod**$-A$, and every module is partially flat and projective with respect to the zero subcategory.

The aim of this paper is to prove that ${}_AN$ (respt. M_A) is partially flat (respt. projective) with respect to the subcategory $\mathscr{X}(A)$ of **Mod**$-A$. (In brief, we write these terms as $\mathscr{X}(A)$–*flat* and $\mathscr{X}(A)$–*projective*.). This is established in Theorem II. In Theorem I a cancellation law related to the objects of $\mathscr{X}(A)$ is proved.

We define $\mathscr{X}(A)$ to be the full additive subcategory of **Mod**$-A$ such that the class of objects of $\mathscr{X}(A)$ is the intersection of the classes of objects of $\mathbf{Mod}(A_M)$, $\mathbf{Mod}(A^N)$, and $\mathscr{D}(A)$, where $\mathbf{Mod}(A_M)$ (respt. $\mathbf{Mod}(A^N)$) is the full additive subcategory of **Mod**$-A$ of all those objects which remain invariant under the composition functor $\mathrm{Hom}_A(M,-)\otimes_B M$ (respt. $\mathrm{Hom}_B(N,-\otimes_A N)$), in a natural way, and $\mathscr{D}(A)$ is the full additive subcategory of **Mod**$-A$ of all those objects on which the two adjoint functors $\mathrm{Hom}_A(M,-)$ and $-\otimes_A N$ remain naturally isomorphic. In above, both bimodules M and N are assumed to be the ingradiants of the Morita context $(A, B, M, N, <,>, [,])$, in which $<,>$ and $[,]$ are (A,A) and (B,B) bimodule homomorphisms: $N\otimes_B M \longrightarrow A$ and $M\otimes_A N \longrightarrow B$, respectively, such that if we set: $<,>(n\otimes m) = <n,m>$ and $[,](m\otimes n) = [m,n]$, then $<n,m>n' = n[m,n']$ and $m<n,m'> = [m,n]m'$.

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In a compatible way the full additive subcategories $\mathbf{Mod}(B_N)$, $\mathbf{Mod}(B^M)$, $\mathscr{D}(B)$ and $\mathscr{X}(B)$ of $\mathbf{Mod}-B$ are defined. Details about $\mathbf{Mod}(A_M)$, $\mathbf{Mod}(A^N)$, $\mathscr{D}(A)$ and $\mathscr{X}(A)$ etc., may be seen in [3] and [4], while for Morita context and various algebraic notions the reader may refer to [2] and [1].

The following cancellation laws (under isomorphisms) hold for the subcategories $\mathscr{X}(A)$ and $\mathscr{X}(B)$.

THEOREM I. *Let N, M, $\mathscr{X}(A)$ and $\mathscr{X}(B)$, be as above and let V and W be objects of the intersecting subcategory $\mathscr{X}(A)$ (respt. $\mathscr{X}(B)$). Then the following cancellation law (under isomorphisms) holds:*

$$V\otimes_A N \cong W\otimes_A N \quad \Rightarrow \quad V\cong W \quad \textit{in } \mathbf{Mod}-A \tag{a}$$

(respt.: $$V\otimes_B M \cong W\otimes_B M \quad \Rightarrow \quad V\cong W \quad \textit{in } \mathbf{Mod}-B\ ).$$

Proof. Being objects of $\mathscr{X}(A)$, V and W are objects of $\mathbf{Mod}(A_M)$ and $\mathscr{D}(A)$. By the definition of $\mathscr{D}(A)$,

$$V\otimes_A N \cong \mathrm{Hom}_A(M,V) \tag{b}$$

via the map

$$\{-,-\}: v\otimes n \longmapsto \{v,n\}: \{v,n\}(m) = v<n,m>. \tag{b$'$}$$

By tensoring both sides of (b) by ${}_BM$, and by the definition of $\mathbf{Mod}(A_M)$, we get the right A-module isomorphisms:

$$V\otimes_A N\otimes_B M \cong \mathrm{Hom}_A(M,V)\otimes_B M \cong V \tag{c}$$

via the canonical maps

$$v\otimes n\otimes m \longmapsto \{v,n\}\otimes m \longmapsto \{v,n\}(m) = v<n,m>. \tag{c$'$}$$

The details about the maps (b′) and (c′) may be seen in [4].

Similarly, we have:

$$W\otimes_A N\otimes_B M \cong \mathrm{Hom}_A(M,W)\otimes_B M \cong W \tag{d}$$

via the canonical maps

$$w\otimes n\otimes m \longmapsto \{w,n\}\otimes m \longmapsto \{w,n\}(m) = w<n,m>.$$

If $V\otimes_A N \cong W\otimes_A N$, then from (c) and (d) we conclude that $V\cong W$ as right A-modules. Hence (a) holds. The remaining part of the theorem holds by symmetry. ∎

THEOREM II. *Let N, M, $\mathscr{X}(A)$ and $\mathscr{X}(B)$ be as above. Then*

(1) ${}_AN$ *is* $\mathscr{X}(A)$*–flat,*

(2) M_A *is* $\mathscr{X}(A)$*–projective,*

(3) ${}_BM$ *is* $\mathscr{X}(B)$*–flat,*

(4) N_B *is* $\mathscr{X}(B)$*–projective.*

Proof. Let the sequence

$$0 \longrightarrow U \xrightarrow{f} V \longrightarrow W \longrightarrow 0 \tag{e}$$

be exact, where $U, V, W \in \mathscr{X}(A)$. We will show that:

$$0 \longrightarrow U \otimes_A N \xrightarrow{f \otimes N} V \otimes_A N \longrightarrow W \otimes_A N \longrightarrow 0 \tag{f}$$

is also exact. The functor $- \otimes_A N$ is right exact in general. We will prove that on $\mathscr{X}(A)$, it is also left exact. Thus we will only show that:

$$U \otimes_A N \xrightarrow{f \otimes N} V \otimes_A N$$

must be injective, or that the $\mathrm{Ker}(f \otimes N) = 0$.

Let $x \in \mathrm{Ker}(f \otimes N)$. We may write $x = \sum u_i \otimes n_i$, where $\sum$ is a finite sum, $u_i \in U$ and $n_i \in N$. Being an object of $\mathscr{X}(A)$, U is also an object of $\mathscr{D}(A)$. So, by (b) and (b′) of Theorem I, the image of x in $\mathrm{Hom}_A(M,U)$ is $\sum \{u_i, n_i\}$. For any arbitrary $m \in M$, by (c) and (c′), we write,

$$\sum \{u_i, n_i\}(m) = \sum u_i <n_i, m> . \tag{g}$$

The image of x in $V \otimes_A N$ under $f \otimes N$ is $\sum f(u_i) \otimes n_i$ and the image of $\sum \{u_i, n_i\}$ under $\mathrm{Hom}(M,f)$ in $\mathrm{Hom}_A(M,V)$ is

$$\mathrm{Hom}(M,f)\left[\sum \{u_i, n_i\}\right] = \sum \{f(u_i), n_i\} .$$

By evaluating this last map on $m \in M$, we get,

$$\begin{aligned} \sum \{f(u_i), n_i\}(m) &= \sum f(u_i) <n_i, m> = \\ &= f\left(\sum u_i <n_i, m>\right) . \end{aligned} \tag{h}$$

By our assumption $(f \otimes N)(x) = 0$. So the corresponding image of x in $\mathrm{Hom}_A(M,V)$, which is $\sum \{f(u_i), n_i\}$, is equal to 0. Since $f \colon U \longrightarrow V$ is injective, by (h), $\sum u_i <n_i, m> = 0$. But then by (g), $\sum \{u_i, n_i\}(m) = 0$. As m is arbitrary, $\sum \{u_i, n_i\} = 0$. Thus it is concluded that $x = 0$ and so $\mathrm{Ker}(f \otimes N) = 0$. Hence (1) holds.

By the definition of $\mathscr{D}(A)$, the functors $\mathrm{Hom}_A(M,-)$ and $-\otimes_A N$ are naturally isomorphic on the subcategory $\mathscr{X}(A)$. Hence (2) holds from (1).

(3) and (4) are satisfied by symmetric properties of the Morita context $(A, B, M, N, <,>, [,])$, and that of the subcategories $\mathscr{X}(A)$ and $\mathscr{X}(B)$. ∎

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