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The Top Cohomology Class of Certain Spaces

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A toplogical space S is rationally elliptic [3] if the spaces $H^*(S;\mathbb{Q})$ and $\pi_*(S)\otimes\mathbb{Q}$ are both finite dimensional. The homogeneous spaces are classical examples of such spaces.

It is known [5, theorem 3] that the (rational) cohomology of a 1–connected elliptic space S is a Poincaré duality algebra.

DEFINITION. A top class of a Poincaré duality algebra $H=\Sigma_{i=0}^{N}H^i$ is a generator of H^N.

In this abstract we present an explicit formula for a cycle representing the top class of certain elliptic spaces, including the homogeneous spaces. For that, we shall relly in the connection between Sullivan's Theory of minimal models and Rational homotopy theory for which [3], [6] and [10] are standar references. Here we recall some notation and conventions:

We shall work over a field $\mathbb{K}$ of characteristic zero unless stated otherwise. A KS–complex is a commutative differential graded algebra (CDGA) $(\Lambda X,d)$ where

$$\Lambda X=\text{Exterior}(X^{odd})\otimes\text{Symmetric}(X^{even})$$

is the free commutative algebra generated by the graded vector space $X=X^{\geq 1}$ which has a well ordered basis $\{x_\alpha\}$ such that $dx_\alpha\in\Lambda X_{<\alpha}$. If $\deg x_\alpha<\deg x_\beta$ implies $\alpha<\beta$, the KS–complex $(\Lambda X,d)$ is minimal. When $(\Lambda X,d)$ is 1–connected $X^1=0$, this is equivalent to say that $dX\subset\Lambda^{\geq 2}X$. Given the CDGA $A(S)$ of differential forms on a topological space S [10], there exists a minimal KS–complex $(\Lambda X,d)$ and a quism (morphism inducing homology isomorphism) $\varphi:(\Lambda X,d)\xrightarrow{\simeq}A(S)$. This is the minimal model of S and is unique up to isomorphism [6, chap. 6]. Since $A(S)$ and $C^*(S;\mathbb{K})$ are connected by a chain of quisms, $H^*(\Lambda X,d)$ and $H^*(S;\mathbb{K})$ are naturally identified.

DEFINITION. A KS–complex $(\Lambda Z,d)$ is said to be a pure tower if $dZ^{even}=0$ and $dZ^{odd}\subset\Lambda(Z^{even})$. If $\dim Z<\infty$, $(\Lambda Z,d)$ is a finite pure tower.

Remark. Homogeneous spaces are examples of spaces whose minimal models are pure towers [4, chap. XI] or [1].

Given a pure tower $(\Lambda Z,d)$ we shall denote $X=Z^{even}$ and $Y=Z^{odd}$. Let $(\Lambda Z,d)=(\Lambda(x_1,\dots,x_n)\otimes\Lambda(y_1,\dots,y_m),d)$ be a finite pure tower and note $f_i=dy_i$, $i=1,\dots,m$. In [5, §3] it is shown that $H^*(\Lambda Z,d)$ is finite dimensional if and only if the algebra

$$\mathbb{K}[x_1,\dots,x_n]/(f_1,\dots,f_m),$$

where $(f_1,\dots,f_m)$ denotes the ideal generated by the polynomials $f_1,\dots,f_m$, is finite dimensional. Then it clear that $m\geqslant n$.

Assume $\dim H^*(\Lambda Z,d)<\infty$ and write:

$$f_i=a_i^1x_1+a_i^2x_2+\cdots+a_i^{n-1}x_{n-1}+a_i^nx_n,\quad i=1,\dots,m,$$

where a_i^j are polynomials in the variables $x_j,x_{j+1},\dots,x_n$. Consider the matrix:

$$A=\begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n-1} & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^{n-1} & a_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_m^1 & a_m^2 & \cdots & a_m^{n-1} & a_m^n \end{bmatrix}$$

and let $P\in\mathbb{K}[x_1,\dots,x_n,y_1,\dots,y_m]$ be the polynomial defined as follows:

$$P=\sum_{1\leqslant i_1<\cdots<i_n\leqslant m}(-1)^{i_1+\dots+i_n}P_{i_1\cdots i_n}\,y_1\cdots\hat{y}_{i_1}\cdots\hat{y}_{i_n}\cdots y_m$$

in which $P_{i_1\cdots i_n}$ is the determinant of the matrix of order n formed by the columns $i_1,\dots,i_n$ of A.

Then we prove:

THEOREM. *P is a cycle representing the top cohomology class of $H(\Lambda Z,d)$.*

EXAMPLES. (1) Consider the homogeneous space $U(4)/(U(2)\times U(2))$. Its minimal model is a finite pure tower of the form [4, chap. XI.4]:

$$(\Lambda(x_2,x_4,y_5,y_7),d),\ dx_2=dx_4=0,\ dy_5=x_2^3-2x_2x_4,\ dy_7=x_4^2-x_2^2x_4$$

in which subscripts denote degrees. Then, we have

$$A = \begin{bmatrix} x_2^2 - 2x_4 & 0 \\ -x_2\, x_4 & x_4 \end{bmatrix}$$

and by theorem above we can compute a generator of the top cohomology class:

$$[x_2^2 x_4 - 2x_4^2] \in H^8(U(4)/(U(2)\times U(2))).$$

(2) Consider now the space $SU(6)/(SU(3)\times SU(3))$ whose minimal model is:

$$(\Lambda(x_4, x_6, y_7, y_9, y_{11}), d),\ dy_7 = -x_4^2,\ dy_9 = -2x_4x_6,\ dy_{11} = -x_6^2.$$

In this case:

$$A = \begin{bmatrix} -x_4 & 0 \\ -2\,x_6 & 0 \\ 0 & -x_6 \end{bmatrix}.$$

Then, $H^{19}(SU(6)/(SU(3)\times SU(3)))$ is generated by $[x_4 x_6 y_9 - 2x_6^2 y_7]$.

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