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Inessential Operators and Incomparability of Banach Spaces

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ABSTRACT

We obtain several characterizations for the classes of Riesz and inessential operators, and apply them to extend the family of Banach spaces for which the essential incomparability class is known, solving partially a problem posed in [6].

Given Banach spaces X and Y, we denote by $L(X,Y)$ the class of all (continuous linear) operators from X to Y. Recall that $T \in L(X,Y)$ is Fredholm ($T \in \Phi(X,Y)$) if it has finite dimensional null space $N(T)$ and finite codimensional closed range $R(T)$.

$S \in L(X)$, X complex Banach space, is a Riesz operator ($T \in \mathfrak{R}(X)$) if for every $0 \neq z \in \mathbb{C}$ we have $zI_X - S \in \Phi$.

$S \in L(X,Y)$ is inessential [8] ($T \in \mathfrak{I}(X,Y)$) if for any $V \in L(Y,X)$ we have $I_X - VS \in \Phi$.

In this paper we give several characterizations of Riesz and inessential operators. Then we apply these results to obtain conditions implying that two Banach spaces are essentially incomparable. In particular, we cover the case in which one of the spaces is subprojective or superprojective in the sense of Whitley [10]. This is the case of $L_p[0,1]$ $(1<p<\infty)$, ℓ_p $(1 \leqslant p < \infty)$, James spaces J and J^*, and Tsirelson spaces T and T^*. Finally we characterize essentially incomparable spaces in terms of Φ.

Inessential operators have been characterized ([8],[9]) as follows

PROPOSITION 1. *For an operator* $T \in L(X,Y)$ *we have*

a) $T \in \mathfrak{I}(X,Y) \Leftrightarrow \dim N(I_X - ST) < \infty$ *for every* $S \in L(Y,X)$.

b) If $\Phi(X,Y)$ *is nonempty, then*

$$T \in \mathfrak{I}(X,Y) \Leftrightarrow \dim N(S-T) < \infty \text{ for every } S \in \Phi(X,Y).$$

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Our first result is a dual version of the above proposition.

PROPOSITION 2. ([2]) *For an operator* $T \in L(X,Y)$ *we have*

a) $T \in \mathfrak{I}(X,Y) \Leftrightarrow \dim Y/\overline{R(I_Y - TS)} < \infty$ *for every* $S \in L(Y,X)$.

b) If $\Phi(X,Y)$ *is nonempty, then*

$$T \in \mathfrak{I}(X,Y) \Leftrightarrow \dim Y/\overline{R(S-T)} < \infty \text{ for every } S \in \Phi(X,Y).$$

The classes Ω_1 and Ω_2, (see [1]), will allow us to give internal characterizations of Riesz and inessential operators. $IS(T)$ will be the class of all infinite dimensional closed invariant subspaces for $T \in L(X)$.

$$\Omega_1(X) := \{T \in L(X) \,/\, M \in IS(T),\ T|_M \text{ isomorphism} \Rightarrow \dim M < \infty\}.$$
$$\Omega_2(X) := \{T \in L(X) \,/\, M \in IS(T),\ M + R(T) = X \Rightarrow \dim Y/M < \infty\}.$$

THEOREM 3. ([2]) *For* $T \in L(X)$, *X a complex Banach space, we have*

a) $T \in \mathfrak{R}(X) \Leftrightarrow T + K \in \Omega_1(X)$ *for every compact* $K \in L(X)$.

b) $T \in \mathfrak{R}(X) \Leftrightarrow T + K \in \Omega_2(X)$ *for every compact* $K \in L(X)$.

THEOREM 4. ([2]) *For an operator* $T \in L(X,Y)$ *we have*

a) $T \in \mathfrak{I}(X,Y) \Leftrightarrow ST \in \Omega_1(X)$ *for every* $S \in L(Y,X)$.

b) $T \in \mathfrak{I}(X,Y) \Leftrightarrow TS \in \Omega_2(Y)$ *for every* $S \in L(Y,X)$.

Two Banach spaces Y and Z are essentially incomparable if $L(X,Y) = \mathfrak{I}(X,Y)$; equivalently, $L(Y,X) = \mathfrak{I}(Y,X)$ [6].

THEOREM 5. ([6]) *The following pairs* Y,Z *are essentially incomparable:*

a) Y *reflexive,* Z *with the Dunford–Pettis property* (DPP).

b) Y *with the reciprocal–*DPP, Z *with the Schur property.*

c) Y *containing no copies of* ℓ_∞, $Z = \ell_\infty, H^\infty$ *or* $C(K)$, K σ*–stonian.*

d) Y *containing no copies of* c_0, $Z = C(K)$.

e) Y *containing no complemented copies of* c_0, $Z = C[0,1]$.

f) Y *containing no complemented copies of* ℓ_1, $Z = L_1(\mu)$.

g) Y,Z *different spaces from* $\{\ell_p\ (1 \leqslant p \leqslant \infty),\ c_0\}$.

In [6] it was proved that given Y,Z two essentially incomparable spaces, any Banach space isomorphic both to a complemented subspace of Y and to a complemented subspace of Z is finite dimensional; and it was conjectured that the converse is true. Note that parts *c),d),e),f)* and *g)* in Theorem 5 support the conjecture. Here we give a positive answer for the case in which one of the spaces is rich in complemented subspaces.

DEFINITION 6. ([10]) A Banach space X is subprojective if every infinite dimensional closed subspace of X contains an infinite dimensional subspace complemented in X.

X is superprojective if any closed infinite codimensional subspace of X is contained in a closed infinite codimensional complemented subspace.

PROPOSITION 7. *a) The following spaces are subprojective:*

1) $c_0(\Gamma)$, $\ell_p(\Gamma)$ $(1 \leqslant p < \infty)$, $L_p[0,1]$ $(2 \leqslant p < \infty)$.

2) The separable (or WCG*) hereditarily*$-c_0$ *spaces.*

3) $C(K)$, *K scattered.*

4) James space J.

5) The original Tsirelson space T^* and its dual T.

6) Hereditarily$-\ell_1$, *non-Schur spaces* X_α *of Azimi–Hagler* [3].

7) Subspaces of subprojective spaces.

b) The following spaces are superprojective:

1) $\ell_p(\Gamma)$ $(1 < p < \infty)$, $L_p[0,1]$ $(1 < p \leqslant 2)$.

2) The dual J^* of James space.

3) The original Tsirelson space T^* and its dual T.

4) Quotients of superprojective spaces.

Proof. For *a1)* and *b1)* see [10]; for *a2)* note that c_0 is always complemented in this case; for *a3)* see [7, th. 11]; for *a4)* and *b2)* see [5, Cor. 11]; for *a5)* and *b3)* see [4, p. 81]; for *a6)* see [3, added in proof]; *a7)* is evident, and *b4)* requires only a moment of thought. ∎

We can describe the space essentially incomparable with one of the above classes as follows.

THEOREM 8. ([2]) *Assume X (or Y) is subprojective or superprojective. Then X and Y are essentially incomparable if and only if any space isomorphic both to a complemented subspace of X and to a complemented subspace of Y is finite dimensional.*

COROLLARY 9. ([2]) *The following pairs of spaces are essentially incomparable:*

1) $\ell_p(\Gamma)$ $(1 \leqslant p \leqslant \infty)$, *$Z$ containing no complemented copy of* ℓ_p.

2) $L_p[0,1]$ $(1 < p < \infty)$, *Z containing no complemented copy of* ℓ_p *or* ℓ_2.

3) $C(K)$, *K scattered; Z containing no complemented copy of* c_0.

4) J or J^*; Z containing no complemented copies of ℓ_2.

5) T^*, Z containing no complemented copy of T^*.

6) X_α (see Prop. 7*), Z containing no complemented copies of ℓ_1.*

Next we characterize essentially incomparable spaces in terms of the class of Fredholm operators in the product space.

THEOREM 10. ([2]) *The following assertions are equivalent:*

a) $L(X,Y)=\mathfrak{I}(X,Y)$; i.e., X and Y are essentially incomparable.

b) For every operator $T=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in L(X\times Y)$ we have

$$T\in\Phi(X,Y) \Leftrightarrow A\in\Phi(X) \text{ and } B\in\Phi(Y).$$

For $T\in\Phi(X,Y)$ we denote the index of T by $\text{ind}(T):=\dim N(T)-\dim Y/R(T)$. Note that the class Φ_n of Fredholm operators with a fixed index is open. We finish the paper with an structural result about this class.

THEOREM 11. ([2]) *Given two essentially incomparable spaces X and Y isomorphic to its respective hyperplanes, $\Phi_n(X\times Y)$ is non connected.*

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