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Fredholm Multipliers of Semisimple Commutative Banach Algebras

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In some recent papers ([1], [2], [3], [4]) we have investigated some general spectral properties of a multiplier defined on a commutative semi–simple Banach algebra. In this paper we expose some aspects concerning the Fredholm theory of multipliers. The main result of [4] states that the Fredholm multipliers of certain commutative semi–simple Banach algebras may be intrinsecally characterized by replacing the so–called Calkin algebra by a quotient algebra of multipliers. We recall that a mapping $T: A \longrightarrow A$, A any commutative Banach algebra with or without a unit is said to be a multiplier if $(Tx)y = x(Ty)$ holds for each $x, y \in A$. In the sequel we shall always suppose that A is semi–simple Banach algebra. We recall that if A is semi–simple then the ideal $\operatorname{soc} A$, the socle of A, does exists ([6]). Let $M(A)$ denote the set of all multipliers of A. $M(A)$ is a closed commutative subalgebra of $L(A)$, the Banach algebra of all bounded linear operators of A. Moreover $M(A)$ is semi–simple ([13], Corollary 1.4.2). Let $K(A)$ be the closed ideal of all compact operators on A and let us denote by $K_M(A)$ the closed ideal $M(A) \cap K(A)$ of $M(A)$. Since $M(A)$ is semi–simple, the socle of A does exist. Moreover for each $T \in K_M(A)$ the spectrum $\sigma_{M(A)}(T) = \sigma(T)$ has 0 as unique accumulation point, so $K_M(A)$ is an inessential ideal of $M(A)$ and hence it is possible to develop an abstract Fredholm theory of $M(A)$ relative to $K_M(A)$ ([6], Chapter F).

Let $\Phi_M(A)$ denote the class of all Fredholm elements of $M(A)$ relative to $K_M(A)$, i.e. those elements of $M(A)$ invertible modulo $K_M(A)$ and let $\Phi(A)$ the set of all Fredholm operators, i.e. the elements of $L(A)$ invertible modulo $K(A)$. Trivially we always have $\Phi_M(A) \subseteq \Phi(A) \cap K(A)$ and this inclusion may be proper [4]. It is of interest the problem of studying conditions on A for which we have $\Phi_M(A) = \Phi(A) \cap K(A)$. This question seems to have an special interest since the Calkin algebra $L(A)/K(A)$ is not commutative however $M(A)$ and

$K_M(A)$ are commutative. Moreover, in several applications there are concrete models of $M(A)$ and $K_M(A)$. We recall that a Banach algebra B is said to be regular if for each closed subset F of the maximal regular closed ideal $\Delta(B)$ and for each $m_0 \notin F$ there exists an element $x \in B$ such that $x\hat{}(F) = \{0\}$ and $x\hat{}(m_0) \neq 0$.

THEOREM 1. ([4]) *Let A be a commutative semi–simple Banach algebra and suppose $M(A)$ regular. Then $\Phi_M(A) = \Phi(A) \cap M(A)$ and any $T \in \Phi(A) \cap M(A)$ has index* 0.

The next theorem shows that the equality $\Phi_M(A) = \Phi(A) \cap M(A)$ holds for a wide class of Banach algebras which includes the case $A = L_1(G)$, G a compact abelian group. Let us denote by $\Phi_+(A)$ the class of all upper semi–Fredholm operators and by $\Phi_-(A)$ the class of all lower semi–Fredholm operators [8]. We recall that a bounded operator T on a Banach space is said to be a Riesz operator if $\lambda I - T$ is a Fredholm operator for each $\lambda \neq 0$ [9].

THEOREM 2. ([4]) *Let $A = \overline{\mathrm{soc}\, A}$. Then $\Phi_M(A) = \Phi(A) \cap M(A)$ and for each $T \in M(A)$ the following are equivalent:*

i) $T \in \Phi_+(A)$.

ii) $T \in \Phi_-(A)$.

iii) $T \in \Phi(A)$.

iv) $T \in \Phi(A)$ *and* $\mathrm{ind}\, T = 0$.

Let $\omega_M(T)$, $W_M(T)$ and $\beta_M(T)$ denote the essential spectrum, the Weyl spectrum and the Riesz spectrum of T relative to the algebra $M(A)$ and to the ideal $K(A)$, respectively. By $\omega(T)$, $W(T)$ and $\beta(T)$ we denote the corresponding sets of the operator T relative to $L(A)$, $K(A)$. Generally we have $\omega_M(T) \subseteq W_M(T) \subseteq \beta_M(T)$ and $\omega(T) \subseteq W(T) \subseteq \beta(T)$.

THEOREM 3. ([4]) *If A is a commutative semi–simple Banach algebra which satisfies either $M(A)$ regular or $A = \overline{\mathrm{soc}\, A}$, then $\omega(T) = W(T) = \beta(T)$ for each $T \in M(A)$.*

In the sequel we shall denote by $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$, $\sigma_{ap}(T)$, the point spectrum, the residual spectrum, the continuous spectrum and the approximate–point spectrum of T, respectively. Let $\Delta(A)$ denote the maximal regular ideal space of A. We have

THEOREM 4. ([4]) *Suppose* $\Delta(A)$ *discrete. Then for each* $T \in M(A)$ *we have* $\sigma_p(T) = \varphi_T(\Delta(A))$ *(the range of the Wang function* φ_T *associated with* T*). Moreover if* $A = \overline{\mathrm{soc}\, A}$ *then* $\sigma_r(T)$ *is empty and* $\sigma(T) = \sigma_{ap}(T)$.

First we recall that a bounded operator on a Banach space is said to be meromorphic if its non zero spectral points are all poles of the resolvent $R(\lambda I - T)^{-1}$ (see [9]).

THEOREM 5. ([4]) *Suppose* $\Delta(A)$ *discrete and* $T \in M(A)$. *The following are equivalent:*

i) T *is a Riesz operator.*

ii) T *is meromorphic.*

iii) $\sigma(T)$ *is a finite set or a sequence which converges to zero.*

iv) $\sigma(T) = \varphi_T(\Delta(A)) \cup \{0\}$, *and* φ_T *vanishes at infinity.*

1) Any semi–simple annihilator Banach algebra has dense socle, (see [7], §32, Corollary 6), and in particular any dual algebra ([12], see also [7]), have this property. Examples of commutative semi–simple dual Banach algebras are $L_p(G)$, G a compact abelian group, $1 \leqslant p \leqslant \infty$, or $C(G)$ the algebra of continuous functions on G with convolution for multiplication ([12], Theorem 15). Thus if T_μ is a convolution operator on $L_p(G)$, defined by $T_\mu(f) = \mu \star f$, $f \in L_p(G)$ and $\mu \in M(G)$, the algebra of all regular Borel measures, the results stated above hold. Let $A = L_1(G)$, G compact and abelian. Then $M(A) \doteq M(G)$ ([15]) and $K_M(A) \doteq L_1(G)$ (see [5]). Hence if $\mu \in M(G)$, T_μ is a Fredholm operator if and only if there exists a $\nu \in M(G)$ and a $\varphi \in L_1(G)$ such that $\mu \star \nu = \delta_0 - \varphi$, where δ_0 is the Dirac measure concentrated at the identity.

2) Let A be a Banach algebra with an orthogonal basis $\{e_k\}$, ([10]). Clearly A has dense socle. Examples of these algebras are ℓ^p, $1 \leqslant p \leqslant \infty$, c_0, and $L_p(\Gamma)$, Γ the torus of $\mathbb{C}$. In this case $M(A)$ is isomorphic to a subalgebra of ℓ^∞ and to any $T \in M(A)$ there corresponds a bounded sequence $(\lambda_k(Te_k))$ ([3]). If $\{e_k\}$ is unconditional, then $M(A) \doteq \ell^\infty$ (see [3]) and $T \in M(A)$ is a Fredholm multiplier if and only if there exists a sequence $\{\nu_k\} \in \ell^\infty$ such that $\nu_k \lambda_k(Te_k) \longrightarrow 1$ ([4]).

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