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On the Free Character of the First Koszul Homology Module [1]

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Let (A,M,K) denote a local noetherian ring A with maximal ideal M and residue field K. Let I be an ideal of A and E the Koszul complex generated over A by a system of generators of I.

The condition: $H_1(E)$ is a free A/I-module, appears in several important results of Commutative Algebra. For instance:

–(Gulliksen [3, Proposition 1.4.9]): The ideal I is generated by a regular sequence if and only if I has finite projective dimension and $H_1(E)$ is a free A/I-module.

–(André [2]): Assume that A is complete intersection. Then, A/I is complete intersection if and only if $H_1(E)^2 = H_2(E)$ and $H_1(E)$ is a free A/I-module.

The purpose of this note is to generalize both results.

We start with a proposition, which is obtained by a slight modification of the proof of [3, Proposition 1.4.9] (see also [5, p. 370]).

PROPOSITION 1. *Let A be a noetherian ring, I an ideal of A, and E the Koszul complex associated to a system of generators of I. Let J be an ideal of A and $\xi \in Z_1(E) \cap JE_1$. If $\mathrm{Tor}^A_{2p}(A/I,A/J) = 0$ for some $p \geqslant 1$, then*

$$\sigma(\xi + B_1(E))^p \in (J+I)/I$$

for all $\sigma \in \mathrm{Hom}_{A/I}(H_1(E),A/I)$.

In particular, if A is local with maximal ideal M, E is associated to a minimal system of generators of I, and I has finite projective dimension, then the trace ideal of the A/I-module $H_1(E)$ is contained in M/I.

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[1] A complete version will appear in the *Journal of Pure and Applied Algebra.*

This result on the trace of $H_1(E)$ allow us to prove the following.

THEOREM 2. *Let* (A,M,K) *be a local noetherian ring,* I *an ideal of* A, *and* E *the Koszul complex associated to a system of generators of* I. *If* $H_1(E)$ *is a free* A/I*-module, then the homomorphism*

$$H_3(A/I,K,K) \longrightarrow H_2(A/I,K,K)$$

is trivial.

Here $H_n(R,S,-)$ stands for the André–Quillen homology of the R-algebra S (see [1]). The homomorphism $H_3(A/I,K,K) \longrightarrow H_2(A/I,K,K)$ is the corresponding to the Jacobi– Zariski sequence [1, Theoreme 5.1] associated to the homomorphisms $A \longrightarrow A/I \longrightarrow K$.

To prove the theorem we consider the M-adic completion of A, which (by Cohen theorem) is a homomorphic image of a regular local ring, and we use several properties of André–Quillen homology (for instance: $H_2(A/I,K,K) = H_1(E) \otimes_{A/I} K$ if E is associated to a minimal system of generators).

Using Theorem 2 we obtain the above mentioned generalizations.

COROLLARY 3. *Let* A *be a complete intersection local ring,* I *an ideal of* A, *and* E *the Koszul complex associated to a system of generators of* I. *The following conditions are equivalent:*

i) A/I *is complete intersection.*

ii) $H_1(E)$ *is a free* A/I*-module.*

COROLLARY 4. *Let* A *be a local noetherian ring,* I *an ideal of* A, *and* E *the Koszul complex associated to a system of generators of* I. *Assume that* $H_1(E)$ *is an* A/I*-free module and that there exists and ideal* J *of finite projective dimension such that* $I \subseteq J$. *Then* I *is generated by a regular sequence.*

To deduce Corollary 4 from Theorem 2 we need to use a strong result of L. Avramov (see [4]).

Finally we get another consequence of Theorem 2, in which not appears explicitely the module $H_1(E)$.

COROLLARY 5. *Let* (A,M,K) *be a local noetherian ring and* I *an ideal of* A. *If* $H_3(A,A/I,K)=0$, *then the homomorphism* $H_3(A/I,K,K) \longrightarrow H_2(A/I,K,K)$ *is trivial.*

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