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On the Schatten $\mathcal{S}_\varphi$ Classes

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The Schatten $\mathcal{S}_p$ classes, $1 \leqslant p < \infty$, were introduced and studied in [6] in connection with the problem of finding suitable classes of operators having a well–defined trace.

In this paper, we consider a generalization $\mathcal{S}_\varphi$ of the Schatten classes $\mathcal{S}_p$ obtained in correspondence with opportune continuous, strictly increasing, sub–additive functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Our purpose is to study the spaces $\mathcal{S}_\varphi$ of the φ–nuclear operators and to compare their properties with those of the well–known space $\mathcal{S}_1$ of nuclear operators. The classes $\mathcal{S}_\varphi$ are subsets of the algebra $\mathscr{L}(\ell^2)$ of all bounded linear operators on ℓ^2. As well known, every compact operator T on ℓ^2 has a representation of the form

$$T = \sum_n \xi_n e_n \otimes f_n \ , \tag{1}$$

where (e_n) and (f_n) are orthonormal systems in ℓ^2 and the sequence (ξ_n) can always be taken to be non–increasing, non–negative and such that $\xi_n \longrightarrow 0$. For $p > 0$, it is customary to denote by $\mathcal{S}_p$ the space of all operators T as in (1) for which the quantity

$$\sigma_p(T) = \sum_n \xi_n^p$$

is finite (cf. [5, §15.5]). Thus, for $1 \leqslant p < \infty$ the $\mathcal{S}_p$ are the Schatten classes while for $0 < p < 1$ the elements of $\mathcal{S}_p$ are the so–called p–nuclear operators (cf. [5, theorem 18.5.2]).

Now, following [3, §II.2], we consider the set Φ' of all continuous, strictly increasing, sub–additive functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that $\varphi(0) = 0$. For any function $\varphi \in \Phi'$ and any scalar sequence $\eta = (\eta_n)$ we put

$$\sigma_\varphi(\eta) = \sum_n \varphi(|\eta_n|)$$

and

$$\ell_\varphi = \{ \eta : \sigma_\varphi(\eta) < \infty \}$$

and we observe that, because of sub–additivity, ℓ_φ is a linear space of sequences on which σ_φ is a metric generating a topology under which $(\ell_\varphi, \sigma_\varphi)$ becomes a complete, metrizable, topological vector space. Since each $\varphi \in \Phi'$ is equivalent to a concave function $\tilde{\varphi} \in \Phi'$ and since $p\varphi \in \Phi'$ whenever $\varphi \in \Phi'$ and $p > 0$, we may always assume that φ is concave and satisfies $\varphi(1) = 1$, so that $\varphi(t) \geqslant t$, for all $t \in [0,1]$. Then, we denote by Φ the set of all such functions and, from now on, we always assume that $\varphi \in \Phi$.

An operator $T \in \mathscr{L}(\ell^2)$ admitting the representation (1) with $(\xi) \in \ell_\varphi$ is called

φ–nuclear and the set of all such operators is denoted by $\mathcal{S}_\varphi$. We observe that, when $\varphi(t)=t^p$ $(0<p\leqslant 1)$, then $\ell_\varphi=\ell^p$ and hence $\mathcal{S}_\varphi=\mathcal{S}_p$, showing that the φ–nuclear operators are a generalization of the p–nuclear ones.

If $T\in\mathcal{S}_\varphi$, we put $\sigma_\varphi(T)=\sigma_\varphi(\xi)$ if $\xi=(\xi_n)$ is the sequence in the representation (1) of T.

THEOREM 1. *$\mathcal{S}_\varphi$ is an operator ideal (in the sense of Pietsch) and σ_φ is a translation invariant metric on it generating a topology under which $\mathcal{S}_\varphi$ becomes a complete, metrizable, topological vector space in which the finite–rank operators are dense. Moreover, the inclusion map $(\mathcal{S}_\varphi,\sigma_\varphi)\longrightarrow(\mathcal{S}_1,\sigma_1)$ is continuous.*

Now we put

$$B_\varphi=\{\,T\in\mathcal{S}_\varphi:\sigma_\varphi(T)\leqslant 1\,\}\ \text{ and }\ B_1=\{\,T\in\mathcal{S}_1:\sigma_1(T)\leqslant 1\,\}.$$

Then we have the following

LEMMA. *B_1 is the closure in $(\mathcal{S}_1,\sigma_1)$ of the absolutely convex hull of B_φ.*

Denote by $\mathcal{S}'_\varphi$ the topological dual of $(\mathcal{S}_\varphi,\sigma_\varphi)$ and put

$$\|A\|_\varphi=\sup\{\,|<T,A>|:T\in B_\varphi\}\ ,$$

for $A\in\mathcal{S}'_\varphi$. Since $\mathcal{S}'_\varphi=(\mathcal{S}_\varphi,\sigma_\varphi)'=(\mathcal{S}_\varphi,\sigma_1)'=\mathcal{S}'_1$, by the lemma, and $(\mathcal{S}'_\varphi,\|\cdot\|_\varphi)=(\mathcal{S}'_1,\|\cdot\|_1)=\mathcal{L}(\ell^2)$, by [6], we have

THEOREM 2. *$(\mathcal{S}'_\varphi,\|\cdot\|_\varphi)$ is a Banach space isometric to $\mathcal{L}(\ell^2)$.*

Turning now our attention to the extreme points of B_φ we find

THEOREM 3. *Let $T\in B_\varphi$. Then the following assertions are equivalent:*
i) T is an extreme point;
ii) $T=e\otimes f$, with $\|e\|=\|f\|=1$.

Because the extreme points of the "unit ball" of $\mathcal{S}_\varphi$ are the same of those of the "unit ball" of $\mathcal{S}_1$, that is the operators of rank 1 and norm 1.

Finally, we investigate the isometries of $(\mathcal{S}_\varphi,\sigma_\varphi)$, i.e. the linear bijections $J:\mathcal{S}_\varphi\longrightarrow\mathcal{S}_\varphi$ such that $\sigma_\varphi(J(T))=\sigma_\varphi(T)$. We find that the results of [1] can be extended to the following

THEOREM 4. *Let $J:\mathcal{S}_\varphi\longrightarrow\mathcal{S}_\varphi$ be linear and onto. The following assertions are equivalent:*
i) J is an isometry;
ii) There exist two unitary operators U,V on ℓ^2 such that $J=U\otimes V$.

Full details and proofs will appear in [7].

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