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Polynomial Characterizations of Banach Spaces not Containing ℓ_1 [1]

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Many properties of Banach spaces can be given in terms of (linear bounded) operators. It is natural to ask if they can also be formulated in terms of polynomial, holomorphic and continuous mappings. In this note we deal with Banach spaces not containing an isomorphic copy of ℓ_1, the space of absolutely summable sequences of scalars.

Throughout, E and F will denote Banach spaces. An operator $T: E \longrightarrow F$ is completely continuous if it takes weakly convergent sequences to convergent ones. A well known result by Odell [10, p. 377] states that E contains no copy of ℓ_1 if and only if every completely continuous operator from E to any F is compact.

Our aim is to provide characterizations of Banach spaces not containing ℓ_1 by means of the nonlinear maps defined on them. For completeness, we include results obtained in [2,4,8].

We write E^* for the topological dual of E. $C(E,F)$ denotes the space of all continuous mappings from E to F . We are interested in two subspaces of $C(E,F)$: the set $C_{wk}(E,F)$ of all mappings taking weakly convergent sequences in E to convergent ones in F, and the set $C_{wb}(E,F)$ of those mappings f which are weakly continuous on bounded subsets, i.e., for any $B \subset E$ bounded, f restricted to B is continuous when B and F are given the restricted weak topology and the norm topology, respectively. Obviously, $C_{wb}(E,F) \subset C_{wk}(E,F) \subset C(E,F)$. For complex E and F, $H(E,F)$ is the space of all holomorphic mappings from E to F. A map $P: E \longrightarrow F$ is a $k-$homogeneous continuous polynomial for a nonnegative integer k if, for each $x \in E$, $P(x) = A(x,\overset{(k)}{\ldots},x)$, where A is a $k-$linear continuous map from $E^k = E \times \overset{(k)}{\ldots} \times E$ to F. $\mathcal{P}(^kE,F)$ stands for the space of such P. The definition and properties of polynomials and holomorphic mappings may be seen in [9]. We shall write $\mathcal{F}_{w\alpha}(E,F) = \mathcal{F}(E,F) \cap C_{w\alpha}(E,F)$ where $\mathcal{F}$ stands for H or $\mathcal{P}$, and α for b or k. Throughout, if the range space is omitted, it is understood to be the complex or the real field $\mathbb{K}$. Thus $\mathcal{P}(^kE) = \mathcal{P}(^kE,\mathbb{K})$.

In recent years, many authors have studied such function spaces; apart from those cited above, the reader is referred to [1,3,5].

THEOREM 1. *The following assertions are equivalent:*

a) E contains no copy of ℓ_1.

b) For every F, $C_{wb}(E,F) = C_{wk}(E,F)$.

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c) If E is complex, for any complex F, $H_{wb}(E,F) = H_{wk}(E,F)$.

d) For every F and positive integer k, $\mathcal{P}_{wb}({}^kE,F) = \mathcal{P}_{wk}({}^kE,F)$.

e) There exists a space F and a integer $k \geqslant 2$ such that $\mathcal{P}_{wb}({}^kE,F) = \mathcal{P}_{wk}({}^kE,F)$.

f) There is an integer $k \geqslant 2$ such that $\mathcal{P}_{wb}({}^kE) = \mathcal{P}_{wk}({}^kE)$.

Sketch of the proof. *a)* $\Longrightarrow$ *b)* may be seen in [8].
b) $\Longrightarrow \dots \Longrightarrow$ *f)* are obvious.
f) $\Longrightarrow$ *a)* Suppose E contains a subspace isomorphic to ℓ_1. Since the natural inclusion $T: \ell_1 \longrightarrow \ell_2$ is absolutely summing, it can be factored through an $L_\infty(\mu)$ [7, p. 60]. By the injectivity of $L_\infty(\mu)$ [7, p. 119], T can be extended to an operator $V: E \longrightarrow \ell_2$. For $x \in E$, write $V(x) = (V_n(x))_{n=1}^\infty \in \ell_2$. For each integer $k \geqslant 2$, define $P_k: E \longrightarrow \mathbb{K}$ by $P_k(x) = \Sigma_{n=1}^\infty (V_n(x))^k$. Easily, $P_k \in \mathcal{P}({}^kE)$. Since V is completely continuous by the Dunford–Pettis property of $L_\infty(\mu)$, it is not difficult to show that $P_k \in \mathcal{P}_{wk}({}^kE)$. Using now [4, Th. 2.9], it can be proved that $P_k \notin \mathcal{P}_{wb}({}^kE)$. ∎

Since $\mathcal{P}_{wk}({}^1E,F)$ is the space of completely continuous operators and $\mathcal{P}_{wb}({}^1E,F)$ is that of compact operators from E to F [5, Prop. 2.5], the last two assertions of the theorem show a different behaviour of polynomials and operators. Indeed, for $k=1$, assertion *f)* is trivially true for every E. Even if F were restricted to be infinite dimensional, the relation *e)* $\Longrightarrow$ *a)* would then fail for $k=1$ since it is known [6, Prop. 3.7] that there exists an infinite dimensional space F such that every completely continuous operator from E to F is compact if and only if E contains no complemented copy of ℓ_1.

The equivalence *a)* $\Leftrightarrow$ *b)* may be found in [8], and *a)* $\Leftrightarrow$ *d)* in [4]. Part *c)* $\Longrightarrow$ *a)* answers a question of [4]; *e)* and *f)* are new.

We say that an operator $T: E \longrightarrow E^*$ is symmetric if it satisfies $\langle y,Tx \rangle = \langle x,Ty \rangle$ for all $x,y \in E$. The following result may be proved:

THEOREM 2. *The following assertions are equivalent:*

a) E contains no copy of ℓ_1.

b) Every completely continuous operator from E to E^* is compact.

c) Every completely continuous symmetric operator from E to E^* is compact.

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