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ON THE CLOSED GRAPH THEOREM BETWEEN TVS AND BANACH SPACES

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We shall use the terminology of [2]. It is well known that the closed graph theorem works between metrizable complete (non necessarily locally convex) spaces. However Functional Analysis has stressed on locally convex spaces. So, $\mathbb{C}$-barrelled spaces is the class of locally convex spaces characterized by verifying the closed graph theorem when Fréchet or Banach spaces are considered in the range.

In topological vector spaces, without convexity conditions, the role of $\mathbb{C}$-barrelled spaces is played, more or less, by $\mathbb{L}$-barrelled spaces. So, a space E is $\mathbb{L}$-barrelled (i.e. one in which each closed string is topological) if, and only if, each linear mapping of E into any (F)-space with closed graph is continuous, [1] and [3]. Every $\mathbb{L}$-barrelled locally convex space is $\mathbb{C}$-barrelled, but the converse is not true in general, so $\mathbb{L}$-barrelled spaces are not a true generalization of $\mathbb{C}$-barrelled spaces.

In [4] we have introduced a class of topological vector spaces, containing $\mathbb{L}$-barrelled spaces, coinciding with $\mathbb{C}$-barrelled spaces when locally convex and verifying the closed graph theorem when the Banach spaces are considered in the range.

DEFINITION. *We shall say a topological vector space E is subbarrelled if each closed absolutely convex string of E is topological.*

Every $\mathbb{L}$-barrelled space is subbarrelled and a locally convex space E is subbarrelled if, and only if, E is $\mathbb{C}$-barrelled. Any non $\mathbb{L}$-barrelled $\mathbb{C}$-barrelled locally convex space is an example of a non $\mathbb{L}$-barrelled subbarrelled space. In [4] we have shown that there exist non $\mathbb{L}$-barrelled subbarrelled spaces which are not locally convex. Therefore, the class of subbarreled spaces is wider than the union of the class of the $\mathbb{L}$-barrelled spaces and the class of the $\mathbb{C}$-barrelled spaces. However, making further assumptions, this may no longer hold.

Let us first recall a fundamental sequence of bounded subsets in the space E is a sequence $\{B_n : n \in \mathbf{N}\}$ of bounded balanced subsets of E such that for each $n \in \mathbf{N}$, $B_n + B_n \subset B_{n+1}$, and each bounded subset of E is contained in some B_n. The following result generalizes the characterization of locally convex $\mathbf{L}$-barrelled spaces given in [2. S19.(2)], to topological vector spaces without local convexity conditions.

THEOREM 1. *Let E be a space with a fundamental sequence of bounded absolutely convex subsets $\{B_n : n \in \mathbf{N}\}$, such that each B_n is a Banach disk. Then E is $\mathbf{L}$-barrelled if, and only if, E is subbarrelled.*

Subbarrelled spaces enjoy good permanence properties. So, the $\mathbf{L}$-inductive limit, the topological direct sum, the separated quotients, the topological product, the subspaces of finite codimension and the completion of subbarrelled spaces are subbarrelled.

If $E(\mathcal{T})$ is a topological vector space, the finest linear topology on E, $\mathcal{T}^f$, is a subbarrelled topology, with $\mathcal{T} \subset \mathcal{T}^f$. Among all the subbarrelled topologies finer than $\mathcal{T}$, there exists a coarsest one, which we shall denote $\mathcal{T}^{st}$, and is subbarrelled since the $\mathbf{L}$-inductive limit of subbarrelled spaces is subbarrelled, holding $\mathcal{T} \subset \mathcal{T}^{st} \subset \mathcal{T}^f$.

Finally, with the help of the following two results we have been able to show that subbarrelled spaces is the class of topological vector spaces characterized by verifying the closed graph theorem when Banach spaces are considered in the range.

PROPOSITION 1. *If A is a continuous linear mapping between the spaces $E(\mathcal{T}_1)$ and $F(\mathcal{T}_2)$, then $A: E(\mathcal{T}_1{}^{st}) \to F(\mathcal{T}_2{}^{st})$ is also continuous.*

PROPOSITION 2. *On a Fréchet space F there exists no strictly coarser Hausdorff subbarrelled topology.*

THEOREM 2. *For a space E the following assertions are equivalent:*

i) E is subbarrelled.

ii) Each linear mapping with closed graph of E into any Fréchet space is continuous.

ii) Each linear mapping with closed graph of E into any Banach space is continuous.

OPEN PROBLEM. *Let E be a subbarrelled space and H any subspace of E of countable codimension. Is H subbarrelled?*

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