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COMPACT AND WEAKLY COMPACT HOMOMORPHISMS BETWEEN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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Recently, many authors have studied compact and weakly compact homomorphisms between function algebras. Among them, Lindström and Llavona [2] treat weakly compact continuous homomorphisms between algebras of type $C(T)$ where T is a completely regular Hausdorff space.

Llavona asked whether the results in [2] are valid in the case of algebras of differentiable functions on Banach spaces. The purpose of this note is to give an affirmative answer to this question, by proving that weakly compact continuous homomorphisms between algebras of differentiable functions are induced by constant mappings. The difficulty we face is that in [2] the existence of continuous functions separating points and closed sets plays an essential role, while in the differentiable case these functions do not exist in general. We deal with Fréchet differentiability, but our results are also valid for Hadamard differentiable functions.

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E and F will be real Banach spaces, and $C^k(E)$ the space of all real valued k-times continuously Fréchet differentiable functions on E. Two topologies are *natural* on $C^k(E)$ (see for instance [3]): the compact open topology of order k and the compact-compact topology of order k. Both topologies coincide if and only if E is finite dimensional. $\mathbb{R}$ will represent the real field.

When we say an algebra homomorphism $A: C^k(E) \longrightarrow C^k(F)$ is continuous, we understand that both $C^k(E)$ and $C^k(F)$ are endowed with one of the *natural* topologies. In [1] it is proved that these non-zero continuous homomorphisms are of the form $Af = f\circ\varphi$ $(f \in C^k(E))$, where $\varphi: F \longrightarrow E$ satisfies $\phi\circ\varphi \in C^k(F)$ for each $\phi \in E^*$.

Following [2], we say a homomorphism $A: C^k(E) \longrightarrow C^k(F)$ is compact [weakly compact] if it maps bounded subsets of $C^k(E)$ into relatively compact [weakly compact] subsets of $C^k(F)$.

1. Proposition. *Let $A: C^k(\mathbb{R}) \longrightarrow C^k(\mathbb{R})$ be a non-zero algebra homomorphism, for some positive integer k, and let $\varphi \in C^k(\mathbb{R})$ be its inducing function. Then the following assertions are equivalent:*

(a) *A has one-dimensional rank;*

(b) *A is compact;*

(c) *A is weakly compact;*

(d) *φ is constant.*

2. Theorem. *Given a positive integer* k, *let* $A: C^k(E) \longrightarrow C^k(F)$ *be a non-zero algebra homomorphism continuous when the algebras* $C^k(E)$, $C^k(F)$ *are both endowed with one of the natural topologies, and let* $\varphi: F \longrightarrow E$ *be its inducing function. Then the following assertions are equivalent:*

(a) *A has one-dimensional rank;*

(b) *A is compact;*

(c) *A is weakly compact;*

(d) *φ is constant.*

However, for $k > m$ there are compact continuous homomorphisms $A: C^k(E) \longrightarrow C^m(F)$ induced by non-constant mappings. This also happens for $k = m = \infty$.

If $k < m$, then every continuous homomorphism $A: C^k(E) \longrightarrow C^m(F)$ is induced by a constant mapping $F \longrightarrow E$ (see [3], 11.2.7).

References.

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[3] LLAVONA, J.G., *Approximation of Continuously Differentiable Functions.* Math. Studies **130**, North-Holland, Amsterdam 1986.