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GENERATING REAL MAPS ON A BIORDERED SET

ANTONIO MARTINON, Departamento Análisis Matemático, Universidad La Laguna
38271 La Laguna (Tenerife), Spain

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If $(X,\leq)$ and $(X,\leq^*)$ are ordered sets, we say that $(X,\leq,\leq^*)$ is a **biordered set** if

(1) $x\leq^* y \Rightarrow x\leq y$.

(2) $y\leq x$ and $z\leq^* x \Rightarrow \exists\ y\wedge z$ and $y\wedge z\leq^* y$,

being $y\wedge z$ the infimum of $\{y,z\}$ for $\leq$. We denote $B(X,\mathbb{R})$ the set of bounded maps of X in $\mathbb{R}$. We define the maps i, s, i^*, s^* on $B(X,\mathbb{R})$, in the following way: for $a\in B(X,\mathbb{R})$ and $x\in X$,

$$ia(x) := \inf_{z\leq x} a(z) , \qquad sa(x) := \sup_{z\leq x} a(z) ,$$

$$i^*a(x) := \inf_{z\leq^* x} a(z) , \qquad s^*a(x) := \sup_{z\leq^* x} a(z) .$$

We can iterate the procedure obtaining many derivated maps from a: i^*sa, ss^*a, s^*is^*sia, ... If a is monotone we only obtain three different new maps.

We will denote a increasing by $a_\uparrow$ and a decreasing by $a^\downarrow$.

Theorem. *Suppose $(X,\leq,\leq^*)$ is a biordered set and $a\in B(X,\mathbb{R})$ is monotone.*

(1) If $a_\uparrow$ then $ia^\downarrow$, $sia_\uparrow$, $i^*a_\uparrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover $ia^\downarrow \leq sia_\uparrow \leq i^*a_\uparrow \leq a_\uparrow$.

(2) If $a^\downarrow$ then $sa_\uparrow$, $isa^\downarrow$, $s^*a^\downarrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover $a^\downarrow \leq s^*a^\downarrow \leq isa^\downarrow \leq sa_\uparrow$.

P r o o f: is based in the following facts:

(1) If $a_\uparrow$ then i^*a, sia, ia are constant on $\{z\in X : z\leq^* x\}$ for every $x\in X$

(2) if $a^\downarrow$ then s^*a, isa, sa are constant on $\{z\in X : z\leq^* x\}$ for every $x\in X$

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The generation process is represented in the following diagrams:

$$a_{\uparrow}\left|\begin{array}{l} sa = a\\ s^*a = a\\ i^*a_{\uparrow}\left|\begin{array}{l} ii^*a = ia\\ si^*a = i^*i^*a= s^*i^*a =i^*a\end{array}\right.\\ ia^{\downarrow}\left|\begin{array}{l} iia = i^*ia =s^*ia = ia\\ sia_{\uparrow}\left|\begin{array}{l} isia= ia\\ ssia= i^*sia =s^*sia=sia\end{array}\right.\end{array}\right.\end{array}\right.$$

$$a^{\downarrow}\left|\begin{array}{l} ia = a\\ i^*a = a\\ s^*a^{\downarrow}\left|\begin{array}{l} ss^*a = a\\ is^*a = s^*s^*a= i^*s^*a =s^*a\end{array}\right.\\ sa_{\uparrow}\left|\begin{array}{l} ssa = s^*sa =i^*sa = sa\\ isa_{\uparrow}\left|\begin{array}{l} sisa= sa\\ iisa= s^*isa =i^*isa=isa\end{array}\right.\end{array}\right.\end{array}\right.$$

The above definition and results are motived by the following facts. We consider an infinite dimensional Banach space, say X. The set of all the closed infinite dimensional subspaces of X, $S(X)$, is a biordered set if we define $M\leq N \iff M\subset N$ and $M\leq^* N \iff M\subset N$ and $dim(N/M)<\infty$.

If T is a linear and continuous operator from an infinite dimensional Banach space X into a Banach space Y, we consider the map

$$n : S(X) \longrightarrow \mathbb{R} \quad ; \quad n(M) := n(TJ_M) := \|TJ_M\| ,$$

where J_M is the injection of M into X and $\|.\|$ denotes the norm. Several authors have defined the following operational quantities:

$$in(T) := \inf_{M\leq X} n(TJ_M) \quad [3]$$

$$i^*n(T) := \inf_{M\leq^* X} n(TJ_M) \quad [1, 4]$$

$$sin(T) := \sup_{M\leq X} in(TJ_M) = \sup_{M\leq X}\ \inf_{N\leq M} n(TJ_N) \quad [3]$$

There are only three differents quantities: in, i^*n, sin [2].

REFERENCES

[1] Lebow, Schechter: *Semigroups of operators and measures of noncompactness.* J. Funct. Anal 7 (1971) 1-26.

[2] Martinon: *Cantidades operacionales en Teoría de Fredholm (Tesis).* Univ. La Laguna (1989).

[3] Schechter: *Quantities related to strictly singular operators.* Indiana Univ. Math. J. 21 (1972) 1061-1071.

[4] Sedaev: *The structure of certain linear operators (In Russian).* Mat. Issled. 5 (1970) 166-175. MR 43#2540; Zbl. 247#47005.