EXTRACTA MATHEMATICAE 5, n.2, 65-67 (1990)

SOME GROTHENDIECK'S PROBLEMS IN THE CONTEXT OF THE α_{pq}-TENSOR PRODUCTS

By
J. A. López Molina y M. J. Rivera Ortún

Departamento de Matemática Aplicada
Universidad Politécnica. 46021 Valencia (España).

Partially supported by the D.G.I.C.Y.T. PS 88-0050

A la memoria de José María García Lafuente

The positive and negative results related to the problem of topologies of Grothendieck [2] have given many information on the projective and injective tensor products of Fréchet and DF-spaces. The pourpose of this paper is to give some results about analogous questions in α_{pq}-Lapresté's tensor products [4, chapitre 1] and in spaces of dominated operators Pietsch [5] for a class of Fréchet spaces having a certain kind of decomposition studied by Bonet and Díaz [1] called T-decomposition. After that we present the main results in order to give positive answers to the following problems:

P1) To localize the bounded sets in $E_{\alpha_{pq}}\hat{\otimes} F$.

P2) If E an F are DF-spaces. Is $E_{\alpha'_{pq}}\otimes F$ a DF-space?

The notation for locally convex spaces is standard. If E is a locally convex space, E_b' denotes its strong dual, $\mathcal{U}_o(E)$ is a basis for the filter of absolutely convex neighbourhoods of zero. Given $U\in\mathcal{U}_o(E)$, $E_{(U)}$ is the corresponding normed quotient space. and P_U the quotient map. If A is a disk in E, E_A represents the linear span of A with the usual normed topology, and $Q_A:E_A\longrightarrow E$ the canonical injection. For every $1\le p\le\infty$, we denote by B_p the unit ball of ℓ_p.

Definition 1.- If α is a tensor norm in the class of normed spaces and E,F are locally convex spaces, the α-topology in $E\otimes F$ is generated by the family of seminorms $\{ \alpha_{U,V}\ ,\ U\in\mathcal{U}_0(E),\ V\in\mathcal{U}_0(F)\}$ such that $\alpha_{UV}(\ z;E\otimes F\ ) = \alpha((P_U\otimes P_V)(z);E_{(U)}\otimes F_{(V)})$ for every $z\in E\otimes F$. We denote by $E\otimes_\alpha F$ the space $E\otimes F$ endowed with the α-topology.

Definition 2.- If $U\in\mathcal{U}_0(E)$, $1\le p\le\infty$, let us consider

$$\ell_p(E) = \{(x_i)\in E^{N}\ ;\ \forall U\in\mathcal{U}_o(E),\ \varepsilon_{pU}((x_i)) = \sup_{x'\in U^0}\Big(\sum_{i=1}^{\infty} |\langle x_i,\ x'\rangle|^p\Big)^{1/p} < \infty \}$$

with the usual modifications for $p=\infty$, endowed with the topology generated by the family of seminorms $\{ \varepsilon_{pU}\ ,\ U\in\mathcal{U}_0(E)\}$

We denote by $\ell_p^o(E)$ the subspace of $\ell_p(E)$ whose elements are limit in $\ell_p(E)$ of finitelly non-zero sequences.

Proposition 1 : If E is a locally convex space then
a) If E is complete, $\ell_p(E)$ is complete. b) $\ell_p^o(E)$ is a closed subspace of $\ell_p(E)$. c) If E is complete then $\ell_p \hat{\otimes}_\varepsilon E$ and $\ell_p^o(E)$ are isomorphics.

Definition 3 : If E and F are locally convex spaces, we define
$\mathcal{A}(E,F) = \{ T \in \mathcal{L}(E,F) \ ; \ \exists\, U \in \mathcal{U}_0(E),\ \exists\, A \text{ diskin } F\ ,\ \exists\, T_0 \in \mathcal{L}(E_{(U)},F_A);$
$\hat{T}_0 \in \mathcal{A}(\hat{E}_{(U)},\hat{F}_A),\quad T = Q_A \cdot T_0 \cdot P_U \}$

We remark that this definition corresponds to the minimal extension of the ideal A to the class of locally convex spaces in the sense of Pietsch [5,29.5] and also to the strong analogous extension of Junek [3,5.3].

Proposition 2 : If F is a quasibarrelled locally convex space, then $(E \otimes_{\alpha'} F)' = \mathcal{A}(E,F'_b)$ algebraically for every locally convex space E.

Now we consider the α_{pq} Lapresté's tensor norms, with $1 < p,q < \infty$ and $1/p + 1/q > 1$. We denote by $\mathcal{A}_{pq}$ the maximal operators ideal associated to α'_{pq} (the Pietsch's (q'p')-dominated operators ideal, see [5, 17.4]). For E,F locally convex spaces, we are going to define a topology in $\mathcal{A}_{pq}(E,F)$. We begin remarking that if E and F are normed spaces and $z \in E \hat{\otimes}_{\alpha_{pq}} F$ there are $(\lambda_i) \in \ell_r$, $(x_i) \in \ell^o_{q'}(E)$, $(y_i) \in \ell^o_{p'}(F)$ such that $z = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$, and
$\alpha_{pq}(z) = \inf \{ \pi_r((\lambda_i))\varepsilon_{q'}((x_i))\varepsilon_{p'}((y_i));\ (\lambda_i) \in \ell_r\ ,\ (x_i) \in \ell^o_{q'}(E),\ (y_i) \in \ell^o_{p'}(F),$
$z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \}$, see Lapresté [4 proposition 1.3]. Then we define

Definition 4.-The BB_{pq} -topology in $\mathcal{A}_{pq}(E,F)$ is the topology generated by the family of seminorms $\{ a_{pqAD}\ ,\ A \in \mathcal{B}(\ell^o_{q'}(E)),\ D \in \mathcal{B}(\ell^o_{p'}(F'_b)) \}$ such that for every $T \in \mathcal{A}_{pq}(E,F)$, $a_{pqAD}(T) = \sup \{ \pi_{r'}((\langle T(x_i),y'_i\rangle))\ ,\ (x_i) \in A\ ,\ (y'_i) \in D \}$

Theorem 1 : If E,F are quasibarreled , then $E \otimes_{\alpha'_{pq}} F$ is a topological subspace of $(\mathcal{A}_{pq}(E'_b,F),\ BB_{pq})$. Moreover if E is reflexive or verifies the density condition then $E'_b \otimes_{\alpha'_{pq}} F$ is a topological subspace of $(\mathcal{A}_{pq}(E,F),BB_{pq})$.

The following definition has been given in [1, def. 1]:

Definition 5 .- A Fréchet space E with an increasing fundamental sequence of seminorms $(\|.\|_k)_{k\in N}$ is called a *decomposable T -space* if for every real sequence $(a_k)_{k\in N}$, $0<a_k\leq 1$ $\forall k\in \mathbb{N}$, we can find a sequence $(P_k)_{k\in N}\subset \mathcal{L}(E,E)$ such that (T1) $P_iP_j = \delta_{ij}P_i$. (T2) $x =\sum_{k\in N} P_k(x)$, $\forall\ x \in E$. (T3) $\|P_k(x)\|_k\leq \|x\|_k$, $\forall x\in E$, $\forall k\in\mathbb{N}$ (T4) $\|P_k(x)\|_{k-1} \leq a_k \|P_k(x)\|_k$, $\forall\ x \in E$, $\forall\ k \in \mathbb{N}$ $k \geq 2$. (T5) $\|.\|_k$ defines the topology induced by E in $P_k(E)$.

Theorem 2.- If E and F are Fréchet decomposable T-spaces and M is a bounded set in $E\hat{\otimes}_{\alpha_{pq}} F$, there exist bounded sets A and D in $\ell^o_{q'}(E)$ and $\ell^o_{p'}(F)$ respectitively such that $M \subset \overline{\Gamma}(H)$, where

$$H=\{z\in E\hat{\otimes}_{\alpha_{pq}} F\ ;\ z=\sum_{i\in N}\lambda_i x_i\otimes y_i;\ (\lambda_i)\in B_r\ ,\ (x_i)\in A,\ (y_i)\in D\}$$

Proposition 4.- If E and F are Fréchet decomposable T-spaces, then the BB_{pq}-topology on $\mathcal{A}_{pq}(E,F'_b)$ coincides with the strong topology in this space corresponding to the duality $(E\hat{\otimes}_{\alpha_{pq}} F)'=\mathcal{A}_{pq}(E,.F'_b)$.

Theorem 3.- If E is a Fréchet decomposable T-space and X a Banach space then $\mathcal{A}_{pq}(E,X)$ with the BB_{pq}-topology is a DF-space.

Proposition 6.- If E is a Fréchet decomposable T-space reflexive or having the density condition, then for every normed space G on verifies that $E'_b \otimes_{\alpha'_{pq}} G$ is a DF-space and it is bornological.

Theorem 5 .- Let E and F be Fréchet decomposable T-spaces. Then $E'_b \otimes_{\alpha_{pq}} F'_b$ is a DF-space.

References

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