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P-ADIC SEMI-FREDHOLM OPERATORS AND MEASURES OF NON-COMPACTNESS

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The p-adic semi-Fredholm theory has been studied by L. GRUSON [1] and J.P. SERRE [8]. One of the basic results of this theory is that for a given Banach space X and a compact operator $T\in L(X)$, then $T-I$ (where I stands for the identity map on X) is a semi-Fredholm operator because it has finite-dimensional kernel and closed range (i.e. $T\in\Phi_+(X)$) and also its range is a finite-codimensional subspace (i.e. $T\in\Phi_-(X)$).

Recently, this kind of operators has been studied in [3] in which the authors give some properties of semi-Fredholm operators related with the preservation of certain classes of closed subsets.

In [4] J. MARTINEZ-MAURICA, TERESA PELLON and the author of this note explore some relationships between semi-Fredholm operators and orthogonality.

Also, semi-Fredholm operators in relation with compact operators are studied in [7].

In this note, we use certain seminorms on the algebra of operators which induce norms on the quotient algebra modulo the ideal of compact operators, to obtain several new descriptions of p-adic semi-Fredholm operators. Proofs of the results included in this note will appear in [5].

Throughout K is a non-archimedean non-trivially valued complete field with valuation $|.|$. X,Y,Z are always going to indicate non-archimedean Banach spaces over K. The spaces of <u>semi-Fredholm</u> <u>operators</u> we will use in the sequel are denoted in the following way,

$$\Phi_+(X,Y) = \{T\in L(X,Y) : R(T) \text{ is closed and } \dim N(T)<\infty\}$$

$$\Phi_-(X,Y) = \{T\in L(X,Y) : R(T) \text{ is closed and } \operatorname{codim} R(T)<\infty\}$$

where for each $T\in L(X,Y)$, $R(T)$ and $N(T)$ denotes, respectively, the range and the kernel of this operator.

A subset A of a normed space X is said to be <u>compactoid</u> if for every $\varepsilon>0$ there exists a finite set $H\subset X$ such that $A\subset\bar{B}_X(0,\varepsilon)+C_0(H)$ (where $\bar{B}_X(0,\varepsilon)=\{x\in X: \|x\|\le\varepsilon\}$ and $C_0(H)$ denotes the absolutely convex hull of H). Also, an operator $T\in L(X,Y)$ is said to be <u>compact</u> if $T(B_X)$ is compactoid (where B_X denotes the open unit ball of X). The space of all compact operators from X into Y will be denoted by K(X,Y).

For operators in $\Phi_+(X,Y)$ the measure of non-compactness we use is the following:

DEFINITION 1 (see [9]). If $T\in L(X,Y)$ we define

$$\nu(T) = \inf\{\|T|D\| : D \text{ is a closed linear subspace of } X \text{ of finite codimension}\}$$

Then, ν is a non-archimedean seminorm on L(X,Y) for which one verifies $\nu(T)=0 \Leftrightarrow T$ is compact ([9], theorem 4.40). This seminorm measures the degree of non-compactness of an operator, since it vanishes precisely on the compact operators.

THEOREM 2. If $T\in L(X,Y)$, then the following properties are equivalent:

i) $T\in\Phi_+(X,Y)$.

ii) For every Banach space Z and for every $S\in L(Z,X)$,

$$T\circ S \text{ compact} \Rightarrow S \text{ compact}.$$

iii) For every infinite-dimensional closed linear subspace of countable type M of X, $T|M$ is not compact.

If in addition X' separates the points of X, properties i)——iii) are equivalent to

iv) There exists a constant $C>0$ such that $\nu(S)\le C\nu(T\circ S)$ for every Banach space Z and for every $S\in L(Z,X)$.

If in addition X is polar (see [6]), then properties i)——iv) are equivalent to

v) $\dim N(T-R)<\infty$ for each $R\in K(X,Y)$.

The characterizations of operators in $\Phi_+(X,Y)$ given in the preceding theorem have a counterpart for operators in $\Phi_-(X,Y)$. The measure of non-compactness we use now is the following:

DEFINITION 3 (see [2]). For each $T\in L(X,Y)$ we define

$$\delta(T) = \inf\{r>0 : \text{there exists a linear subspace } F \text{ of } Y,\ \dim F<\infty,\ T(B_X) \subset F+\bar{B}_Y(0,r)\}.$$

Then δ is a non-archimedean seminorm on L(X,Y) for which one verifies $\delta(T)=0 \Leftrightarrow T$ is compact ([7], theorem 3.3) and hence again δ measures the degree of non-compactness of an operator.

THEOREM 4. For a $T\in L(X,Y)$ we consider the following properties:

i) $T\in\Phi_{-}(X,Y)$.

ii) There exists a constant $C>0$ such that $\delta(S)\leq C\delta(S\circ T)$ for each Banach space Z and each $S\in L(Y,Z)$.

iii) For each Banach space Z and for each $S\in L(Y,Z)$,

$$S\circ T \text{ compact} \Rightarrow S \text{ compact}.$$

iv) If M is a closed linear subspace of Y with infinite codimension then $Q_M\circ T$ is not compact (where $Q_M:Y\longrightarrow Y/M$ is the quotient map).

v) $\operatorname{codim}\overline{R(T-H)}<\infty$ for each $H\in K(X,Y)$.

Then, one verifies i)⇒ii)⇒iii)⇒iv)⇒v).

If in addition X and Y satisfy one of the following properties:

(I) X is polar and Y is strongly polar (see [6])

or

(II) X,Y are reflexive and X',Y' are strongly polar,

we have that properties i)——v) are equivalent.

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