EXTRACTA MATHEMATICAE 5, n.1, 18-20 (1990)

RATIONAL FIBRATIONS IN DIFFERENTIAL HOMOLOGICAL ALGEBRA

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1980 A.M.S. Classification 55P62, 18G15.

The vector space $Ext_{C^*(S;\mathbf{K})}(\mathbf{K}, C^*(S;\mathbf{K}))$, in which S is a 1–connected topological space, turns out to be a nice homotopy invariant whose study was developed from the analogous concept in local algebra, $Ext_R(\mathbf{K}, R)$ for a local commutative ring R [2]. This invariant can be thought as the reduced homology of a "virtual Spivak fibre" and is a key fact in the establishment of interesting results on others, more classical, homotopy invariants [6].

Given a fibration $F \to E \to B$ of simply connected spaces in which $H^*(B;\mathbf{Q})$ has finite type (i.e., it is finite dimensional in each degree) and $H^*(F;\mathbf{Q})$ is finite dimensional, a result of Y. Felix, S. Halperin and J.C. Thomas [6,theorem 4.3] asserts that:

$$Ext_{C^*(F;\mathbf{Q})}(\mathbf{Q}, C^*(F;\mathbf{Q}))\widehat{\otimes} Ext_{C^*(B;\mathbf{Q})}(\mathbf{Q}, C^*(B;\mathbf{Q}))$$

$$\cong Ext_{C^*(E;\mathbf{Q})}(\mathbf{Q}, C^*(E;\mathbf{Q})).$$

This statement was proved via duality and, since it is the result of composing several isomorphisms, it is not easy to work with to develope explicit applications. Our purpose is to generalize this result and give a natural and explicit way of constructing this isomorphism. We begin by remarking two algebraic facts:

1. Let R be a DGA and let M, N, U be R-modules. Define a natural map,

$$\psi : Ext_R(M,N) \otimes Ext_R(U,M) \longrightarrow Ext_R(U,N),$$

as follows: choose semifree resolutions $P \xrightarrow{\simeq} M$ and $Q \xrightarrow{\simeq} U$ of M and U respectively and let $f : Q \to M$, $g : P \to N$ represent $[f] \in Ext_R(U,M)$ and $[g] \in Ext_R(M,N)$. Then, define

$$\psi([g] \otimes [f]) = [g \circ \overline{f}],$$

in which $\overline{f}$ is the homotopy lifting of f to P:

$$\begin{array}{ccc} & & P \\ & \overset{\overline{f}}{\nearrow} & \downarrow\simeq \\ Q & \xrightarrow{f} & M \end{array}$$

2. On the other hand, given a sequence of DGA's morphisms, $R \to S \to T$, there are two natural transformations, defined as in the classical way:

$$\eta : Ext_R(-,-) \longrightarrow Ext_S(- \otimes_R S; - \otimes_R S),$$

$$\nu : Ext_T(-,-) \longrightarrow Ext_S(-,-).$$

Next, we combine 1. and 2. as follows: Let $A \to (A \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ be a KS-extension of the commutative differential graded algebra (CDGA) A. Given an A-module N and a $(\Lambda Y, \overline{d})$-module U we define a map:

$$\varphi : Ext_A(\mathbf{K}, N)\widetilde{\otimes} Ext_{\Lambda Y}(U, \Lambda Y) \longrightarrow Ext_{A\otimes\Lambda Y}(U, N \otimes \Lambda Y)$$

by $\varphi(\sum_{-\infty}^{\infty} \alpha_i \otimes \beta_i) = \sum_{-\infty}^{\infty}(\eta(\alpha_i) \circ \nu(\beta_i))$.

Observe that φ is well defined. In fact, if $\nu(\beta_i)$ is represented by $f_i : P \to (\Lambda Y, \overline{d})$ with $P \xrightarrow{\simeq} U$ semifree resolution, for any $n \in \mathbf{Z}$ there is an s such that $|f_i| < n$ if $i \geq s$ (by definition of semicomplete tensor product). Therefore, given $\Phi \in P$ there is only a finite number of i for which $f_i(\Phi) \neq 0$.

Our aim is to prove:

THEOREM A: *Let $(A \otimes \Lambda Y, d)$ be a KS-extension of the connected CDGA A. Let N be an A-module and let $U = U^{\geq r}$, for some $r \in \mathbf{Z}$, be a $(\Lambda Y, \overline{d})$-module of finite type. If $H^*(\Lambda Y, \overline{d})$ is finite dimensional, then:*

$$\varphi : Ext_A(\mathbf{K}, N)\widehat{\otimes} Ext_{\Lambda Y}(U, \Lambda Y) \xrightarrow{\cong} Ext_{A\otimes\Lambda Y}(U, N \otimes \Lambda Y)$$

is an isomorphism.

We prove that there is another set of conditions which makes φ an isomorphism:

For any minimal KS-extension $(A \otimes \Lambda Y, d)$ in which A or $(\Lambda Y, \overline{d})$ is 1-connected and $H^*(\Lambda Y, \overline{d})$ has finite type, there is another KS-extension $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ and quisms α and γ, such that the diagram,

$$\begin{array}{ccccc} A & \longrightarrow & (A \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \overline{d}) \\ \alpha \uparrow \simeq & & \gamma \uparrow \simeq & & \| \\ (\Lambda X, d) & \longrightarrow & (\Lambda X \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \overline{d}) \end{array}$$

commutes, and $(\Lambda X, d)$ is the minimal model of A [7,§6]. If $(\Lambda X \otimes \Lambda Y, d)$ is minimal, we shall say that $(A \otimes \Lambda Y, d)$ is an *intrinsic* KS-extension [9,def.4.13].

Then, we prove:

THEOREM B: *Let $(A \otimes \Lambda Y, d)$ be an intrinsic KS-extension and let $N = N^{\geq r}$ be an A-module. If Y is finite dimensional, then,*

$$\varphi : Ext_A(\mathbf{Q}, N) \otimes Ext_{\Lambda Y}(\mathbf{Q}, \Lambda Y) \xrightarrow{\cong} Ext_{A\otimes\Lambda Y}(\mathbf{Q}, N \otimes \Lambda Y)$$

is an isomorphism.

Now, given a fibration $F \to E \xrightarrow{p} E$ of simply connected spaces, we can consider its associated sequence of differential forms $A(B) \to A(E) \to A(F)$, [12] or [8,chap.20]. A classical result on Sullivan's theory of minimal models [9,§4] asserts the existence of a KS-extension $A(B) \to (A(B) \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$ and quisms ϕ and α, such that the following diagram commutes:

$$\begin{array}{ccccc} A(B) & \longrightarrow & A(E) & \longrightarrow & A(F) \\ \| & & \phi \uparrow \simeq & & \simeq \uparrow \alpha \\ A(B) & \longrightarrow & (A(B) \otimes \Lambda Y, d) & \longrightarrow & (\Lambda Y, \overline{d}) \end{array}$$

This KS-extension is intrinsic if and only if $\pi_*(\rho) \otimes \mathbb{Q}$ is surjective [9,§4]. It is also known [3,chap.11] that the vector space Y can be identified to the rational homotopy of the fibre, $\pi_*(F) \otimes \mathbb{Q}$. On the other hand, for any space S, the DGAs $A(S)$ and $C^*(S;\mathbb{Q})$ have the same weak homotopy type.

Considering these facts, and applying theorems A and B to the KS-extension $A(B) \to (A(B) \otimes \Lambda Y, d) \to (\Lambda Y, \overline{d})$, we deduce:

THEOREM C: *Let $F \to E \xrightarrow{\rho} B$ be a fibration of simply connected spaces in which B has finite $\mathbb{Q}$-type, and where either (i) $H^*(F;\mathbb{Q})$ is finite dimensional or (ii) $\pi_*(F) \otimes \mathbb{Q}$ is finite dimensional and $\pi_*(\rho) \otimes \mathbb{Q}$ is surjective. Then, there exists an isomorphism:*

$$\varphi : Ext_{C^*(B;\mathbb{Q})}(\mathbb{Q}, C^*(B;\mathbb{Q})) \widehat{\otimes} Ext_{C^*(F;\mathbb{Q})}(\mathbb{Q}, C^*(F;\mathbb{Q}))$$
$$\xrightarrow{\cong} Ext_{C^*(E;\mathbb{Q})}(\mathbb{Q}, C^*(E;\mathbb{Q})).$$

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