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APPROXIMATION OF CONVEX BODIES BY POLYNOMIAL BODIES III: AREA CASE

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With this paper we conclude one part of a work, initiated in [3] and [4], concerning the approximation of planar symmetric convex bodies by polynomial bodies. We use the notations of [3] and [4], and we refer the reader to these papers whenever definitions are omitted, and to see the basic scheme of the problem. (See the above *extracta* of the author in this journal).

1.- We can read in Day [1](1947): "Loewner has shown (in some unpublished work) that there exists a unique ellipse E of minimal area circumscribed about S, and that this ellipse touches S in at least four points...". We have generalized this result in [3] and [4] for the width and radius cases, and we now do this for the area case. First, a simple consequence of the strict convexity of the function $D_a(\mathbf{P})=m[B_P]$, proves the

THEOREM (1). There exists a unique $\mathbf{P}\in\mathcal{P}^e(B)$ such that $m[B_P]<m[B_Q]$, $\mathbf{Q}\in\mathcal{P}^e(B)$.

THEOREM (2). Let $B_P\in\mathcal{B}_a^e(B)$ $[B_P\in\mathcal{B}_a^i(B)]$. Then $S_P\cap S$ has at least 2k+2 points.

Now, for the area criterion we have a complementary result of (2), not true for the width and radius criteria. Let $B_P\in\mathcal{B}_a^e(B)$ and $C=|S_P\cap S|$, the convex envelope of $S_P\cap S$-. We have that $C\in\mathcal{C}_s$ and $\mathbf{P}\in\mathcal{P}^e(C)$, because $C\subset B\subset B_P$, and

THEOREM (3). $B_P\in\mathcal{B}_a^e(C)$.

COROLLARY (4). Let E be the ellipse of Loewner of minimal area circumscribed about B. If $E\cap S=\{x,y,-x,-y\}$, then x and y are mutually Birkhoff orthogonal.

2.- Now we define the best approximation area-exterior operator. For each $B\in\mathcal{C}_s$, there exists a unique $\mathbf{P}\in\mathcal{P}^e(B)$ such that $B_P\in\mathcal{B}_a^e(B)$. So we can define the operator $\mathcal{A}_e:\mathcal{C}_s\longrightarrow\mathcal{P}_{2k}$, such that $\mathcal{A}_e(B)=\mathbf{P}$, and $B_P\in\mathcal{B}_a^e(B)$.

THEOREM (5). Let $C, C_n\in\mathcal{C}_s$, with $n\in\mathbb{N}$. If $C_n\to_H C$, then: a) $\mathcal{A}_e(C_n)=\mathbf{P}_n\to\mathcal{A}_e(C)=\mathbf{P}$. b) $m[B_{P_n}]\to m[B_P]$.

3.-We ask now whether the contact and draw-back theorems given by us for the three approximation criteria are the best possible. In this sense we see that, for all $k\in\mathbb{N}$, there exist convex bodies $B\in\mathcal{C}_s$ such that the sphere of the best approximation polynomial body -between polynomials $\mathbf{P}\in\mathcal{P}^e_{2k}(B)$- touches the sphere S of B in 2k+2 points exactly (for the three criteria), and moves away from S in 2k+2 points too (for the width and radius cases).

THEOREM (6). Let $B\in\mathcal{C}_s$ be a regular 2m-gon in $\mathbb{R}^2$ centered at the origin. If $\mathbf{P}\in\mathcal{P}^e_{2k}(B)$, with $1\leq k\leq m-1$, and B_P is the exterior best approximation -between polynomials of degree 2k-, with the width, radius or area criteria, then S_P is the circumscribed circumference to B.

With this result we can give a negative answer (for the best approximation of sets in $\mathcal{C}_s$ by polynomial bodies) of a pretty result due to Dowker [2] (1944) that established the following: "If M_n means the n-gon of minimum area circumscribed around a convex region $\mathcal{R}$ in the plane, then the area of M_n is a convex function of n". This is to say $2m[M_n]\leq m[M_{n-1}]+m[M_{n+1}]$. In our case we raise a natural question: if $\mathbf{P}_n\in\mathcal{P}^e_{2n}(B)$ and $B_{P_n}\in\mathcal{B}^e_a(B)$ -between polynomials of degree 2n-, does the following inequality hold: $2m[B_{P_n}]\leq m[B_{P_{n-1}}]+$ $+m[B_{P_{n+1}}]$?. The answer is negative.

5.-Although for width and radius cases, the unicity of best approximation was a simple consequence of the contact and draw-back theorems, in the area-interior case the situation is different. We don't know whether or not there exists a proof for it, even though we suspect that there is. We must content ourselves with a (apparently) weak result, although we will have, as a consequence, the continuity theorem if the unicity of best approximation were proved.

THEOREM (7). Let $C,C_n\in\mathcal{C}_s$, with $n\in\mathbb{N}$. If $B_{P_n}\in\mathcal{B}^i_a(C_n)$, for each $n\in\mathbb{N}$, and $C_n\to_H C$, then: a) $\mathbf{P}_n$ is bounded (then $\langle\overline{\mathbf{P}_n}\rangle\neq\emptyset$), b) If $\mathbf{P}\in\langle\overline{\mathbf{P}_n}\rangle$, then $B_P\in\mathcal{B}^i_a(C)$, c) If $B_P\in\mathcal{B}^i_a(C)$, then $\lim m[B_{P_n}]=m[B_P]$.

6.-We have the unicity of best approximation area-interior, of a set $B\in$ $\in\mathcal{C}_s$, when we use polynomials $\mathbf{P}\in\mathcal{P}^i_2(B)$ -it is the simple case of an ellipse of Loewner-. But we don't know whether this result is true for all $n\in\mathbb{N}$, as we want. Nevertheless we can give a new approach to this question with a result in the way that authors as [6] Kenderov(1980), and [5] Gruber & Kenderov (1982) introduced in the problem of the approximation of convex bodies by polygons. In this problem, the unicity of best approximation area interior (or exterior), by n-gons, is trivially not true in general. But they observed that the convex sets, in which the unicity is not true, are few in relation to all the convex sets. Because they form a set of first category with the locally compact topology of the Hausdorff metric in $\mathcal{C}_s$.

THEOREM (8). $\mathcal{C}_{nk}=\{C\in\mathcal{C}_s: \exists B_P,B_Q\in\mathcal{B}_a^1(C); P,Q\in\mathcal{P}_{2k}^1(C), m[B_P\Delta B_Q]\geq 1/n\}$ are closed with empty interior. Then $\mathcal{C}_u=\mathcal{C}_s-\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\mathcal{C}_{nk}$ is dense in $\mathcal{C}_s$, and for all $B\in\mathcal{C}_u$ we have the unicity of best approximation area-interior for all $k\in\mathbb{N}$.

7.-We will see that for the interior approximation we have a "dual" result of (3), which we call the "caged amoeba" theorem, because loosely speaking, it says: "The greatest *amoeba* (polynomial body), of 2k *pseudopodia* (of degree 2k), confined into an oval enclosure, cannot fatten any more -however much she tries-, even if we increased the enclosure by placing straight walls at the spots where the *amoeba* rested before".

THEOREM (9). Let $k\in\mathbb{N}$ such that $\cup[\mathcal{C}_{nk}:n\in\mathbb{N}]=\emptyset$. If $B\in\mathcal{C}_s$, $B_P\in\mathcal{B}_a^1(B)$, with $P\in\mathcal{P}_{2k}^1(B)$, and $B_t\in\mathcal{C}_s$ is the convex set defined by the tangents to B at the points of $S_P\cap S$, then $B_P\in\mathcal{B}_a^1(B_t)$.

COROLLARY (10). If E is the ellipse of Loewner of maximal area inscribed into $B\in\mathcal{C}_s$ and $E\cap B=\{x,y,-x,-y\}$, then x and y are Birkhoff orthogonal.

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