EXTRACTA MATHEMATICAE 4, n.3, 154-156 (1989)

NONEXISTENCE OF NONTRIVIAL SOLUTIONS TO SOME NONLINEAR VOLTERRA EQUATIONS.

W. Okrasinski

Institute of Mathematics. Wroclaw University. Pl. Grunwaldski 2-4. 50-384 Wroclaw. Poland.

AMS (1980) Class Number: 45D05, 45G10

The nonlinear Volterra integral equations

$$u(x)=\int_0^x k(x-s)g(u(s))ds \quad (x>0), \tag{1}$$

where k is a nonnegative function such that $\int_0^x k(s)ds>0$ for $x>0$ and g is concave increasing function such that $g(0)=0$, in some physical problems appears (see |4|). With respect to a physical meaning of (1) we see for nontrivial solutions to (1). A function u is the nontrivial solution if it is continuous and positive on an interval $[0,\delta]$ $(\delta>0)$. Let us note that (1) has trivial solution $u=0$. Nontrivial solutions can be found not for all k and g. Some sufficient and necessary conditions for the existence of nontrivial solutions to (1) can be found in Gripenberg's paper [3] and its extensions (see |1|, |4|, |5|). In physical applications the function g has usually the form $n^p (p\in(0,1))$ (see |4|). In the case of such g sufficient conditions presented in |4| gnarantee the existence of nontrivial solutions for kernels $k(x)\geq C\exp(-\frac{1}{x^\alpha})(\alpha\in(0,1),C>0)$ On the other side it is known that nontrivial solutions exist for functions k satisfying this last condition for any $\alpha>0$. (see |2|). Here we want to find such kernels k for which equation (1) with $g(u)=u^p (p\in(0,1))$ has not nontrivial solutions. we can show the following lemma.

Lemma. Let $k:[0,1]\longrightarrow[0,+\infty)$ be an increasing absolutely continous function such that $k(x)>0$ for $x>0$ and $g:[0,+\infty)\longrightarrow[0,+\infty)$ be a concave increasing continuous function such that $g(0)=0$ and $g(u)/u \longrightarrow +\infty$ as $u\longrightarrow 0+$. If equation (1) has a nontrivial solution u then u^{-1} exists and is a concave increasing function. Moreover

$$u^{-1}(x)\geq K^{-1}(x)+u^{-1}(g^{-1}(x)) \tag{2}$$

on an interval $[0,\delta]$ $(\delta>0)$ (K^{-1} denotes the inverse function to $K(x)=\int_0^x k(s)ds$).

Corollary. If equation (1) has a nontrivial solution u then

$$u^{-1}(x)\geq \sum_{n=0} K^{-1}(g^{-1})^n(x))$$

for $x\in[0,\delta]$ $(\delta>0)$, where $(g^{-1})^n$ denotes the superposition of g^{-1} n times, $(g^{-1})^o(x)=x$, etc. To prove (3) we successively apply mequality (2). In the m-th step we get

$$u^{-1}(x)\geq \sum_{n=o}^{m-1} K^{-1}((g^{-1})^n(x))+u^{-1}((g^{-1})^m(x)) \quad (x\in[0,\delta])$$

Since u^{-1} is increasing continuous such that $u^{-1}(0)=0$ and g^{-1} is convex such that $g^{-1}(x)\leq x$ then $u^{-1}((g^{-1})^m(x))\rightrightarrows 0$ on $[0,\delta]$ as $m\longrightarrow\infty$. The Corollary is proved.

Example. Let $K(x)=\exp(-\exp\frac{1}{x^\alpha})$ $(\alpha\geq 1)$. We consider the equation

$$(4) \qquad u(x)=\int_0^x k(x-s)[u(s)]^p ds \quad (p\in(0,1))$$

with the k=K'. We can easy compute that $K^{-1}(x)=1/[\log\log(1/x)]^{\frac{1}{\alpha}}$ $g^{-1}(x)=x^{1/p}$ and $(g^{-1})^n(x)=x^{(1/p)^n}$. We get

$$K^{-1}((g^{-1})^n(x))=1/[n\log(\tfrac{1}{p})+\log\log(\tfrac{1}{x})]^{\frac{1}{\alpha}}$$

If equation (4) has a nontrivial solution u then by (3) we get

$$(5) \qquad u^{-1}(x)\geq \sum_{n=o}^{\infty} 1/[n\log(1/p)+\log\log(1/x)]^{1/\alpha}$$

for $x\in[0,\delta]$. But if we fix $x\in[0,\delta]$ then we see the series in (5) is divergent to $+\infty$ because of its equivalence with the series $1/n^{1/\alpha}$ $(\alpha\geq 1)$. It means equation (4) has not nontrivial solutions.

This last example shows that keonels k satisfying nequality $k(x)\leq C\,[\exp-\exp(1/x^\alpha)|$ $(c>0, \alpha\geq 1)$ have not any physical meaning.

REFERENCES

|1| P.J. Bushell, W. Okrasinski. Uniqueness of solutions for a class of nonlinear Volterra equations with convolution kernel. Math. Proc. Camb. Phil. Soc. 106 3(1989).

|2| P.J. Bushell, W. Okrasinski. Nonlinear Volterra integral equation with convolution kernel (to apper in J. London. Math. Soc.).

|3| G. Gripenberg. Unique solutions of some Volterra integral equations. Math. Scand. 48(1981).

|4| W. Okrasinski. Nonlinear Volterra equations and physical applications. Appendix. Extracta Math. 4. 2(1989).

|5| W. Okrasinski. Remarks on nontrivial solutions to a Volterra integral equation (to appear in Math. Meth. in the Appl. Sci.).